

Appendices

Appendix A

Cesium D Line Data

A.1 Overview

In this appendix we review many of the physical and optical properties of cesium that are relevant to the experiments in this dissertation. In particular, we give parameters that are useful in treating the mechanical effects of light on cesium atoms. The atomic parameters for the D_1 and D_2 optical transitions are given, although only the D_2 transition values are relevant to the work in this dissertation. The measured numbers are given with their original references, and the calculated numbers are presented with an overview of their origin along with references to more comprehensive discussions of their underlying theory.

A.2 Cesium Physical and Optical Properties

Some useful fundamental physical constants are given in Table A.1. The values given are the 1998 CODATA recommended values, as listed in [Mohr00]. Some of the overall physical properties of cesium are given in Table A.2. Cesium has 55 electrons, only one of which is in the outermost shell. ^{133}Cs is the only stable isotope of cesium, and is the only isotope we consider in this reference. The mass is taken from the high-precision measurement of [Bradley99], and the density, melting point, boiling point, and heat capacities are taken from [Lide00]. The vapor pressure at 25°C and the vapor pressure curve in Fig. A.1 are taken from the vapor pressure model given by [Nesmeyanov63], which is

$$\begin{aligned} \log_{10} P_v &= -219.482\,00 + \frac{1088.676}{T} \\ &\quad - 0.083\,361\,85\,T + 94.887\,52\,\log_{10} T \quad (\text{solid phase}) \\ \log_{10} P_v &= 8.221\,27 - \frac{4006.048}{T} \\ &\quad - 0.000\,601\,94\,T - 0.196\,23\,\log_{10} T \quad (\text{liquid phase}), \end{aligned} \tag{A.1}$$

where P_v is the vapor pressure in torr, and T is the temperature in K. The ionization limit is the minimum energy required to ionize a cesium atom; this value is taken from Ref. [Weber87].

The optical properties of the cesium D line are given in Tables A.3 and A.4. The properties are given separately for each of the two D-line components; the D₂ line (the $6^2S_{1/2} \rightarrow 6^2P_{3/2}$ transition) properties are given in Table A.3, and the optical properties of the D₁ line (the $6^2S_{1/2} \rightarrow 6^2P_{1/2}$ transition) are given in Table A.4. Of these two components, the D₂ transition is of much more relevance to current quantum and atom optics experiments, because it has a cycling transition that is used for cooling and trapping cesium. The frequencies ω_0 of the transitions were measured using an optical frequency comb [Udem00; Udem99]; the vacuum wavelengths λ and the wave numbers k_L are then determined via the following relations:

$$\lambda = \frac{2\pi c}{\omega_0} \quad k_L = \frac{2\pi}{\lambda} . \quad (\text{A.2})$$

The air wavelength $\lambda_{\text{air}} = \lambda/n$ assumes index of refraction of $n = 1.000\,268\,21$, corresponding to dry air at a pressure of 760 torr and a temperature of 22°C. The index of refraction is calculated from the Edlén formula [Edlén66]:

$$n_{\text{air}} = 1 + \left[\left(8342.13 + \frac{2\,406\,030}{130 - \kappa^2} + \frac{15\,997}{38.9 - \kappa^2} \right) \times \left(\frac{0.001\,388\,23\,P}{1 + 0.003\,671\,T} \right) - f(5.722 - 0.0457\kappa^2) \right] \times 10^{-8} . \quad (\text{A.3})$$

Here, P is the air pressure in torr, T is the temperature in °C, κ is the vacuum wave number $k_L/2\pi$ in μm^{-1} , and f is the partial pressure of water vapor in the air, in torr. This formula is appropriate for laboratory conditions and has an estimated uncertainty of $\leq 10^{-8}$. The lifetime is an average of two recent measurements; the first [Rafac99] used a fast beam laser technique, yielding lifetimes of 35.07(10) ns for the $6^2P_{1/2}$ state and 30.57(7) ns for the $6^2P_{3/2}$ state, while the second [Young94] used a photon-counting method, giving lifetimes of 34.75(7) ns ($6^2P_{1/2}$) and 30.41(10) ns ($6^2P_{3/2}$) state. The former measurement is taken to supersede several measurements by some of the same experimenters using the same technique [Rafac94; Tanner92], and another measurement of comparable quoted uncertainty (29.9(2) ns for the $6^2P_{3/2}$ state) [Rydberg72] is excluded because of a substantial disagreement with all recent precision measurements [Rafac99]. Another precise measurement of the ratios of the D₁ and D₂ transition strengths [Rafac98] was not accounted for in the values quoted here. A general discussion of precision lifetime measurement methods can be found in [Tanner95]. Inverting the lifetime gives the decay rate, which is also the natural (homogenous) line width of the emitted radiation.

The recoil velocity v_r is the change in the cesium atomic velocity when absorbing or emitting a resonant photon, and is given by

$$v_r = \frac{\hbar k_L}{m} . \quad (\text{A.4})$$

The recoil energy $\hbar\omega_r$ is defined as the kinetic energy of an atom moving with velocity $v = v_r$, which is

$$\hbar\omega_r = \frac{\hbar^2 k_L^2}{2m} . \quad (\text{A.5})$$

The Doppler shift of an incident light field of frequency ω_L due to motion of the atom is

$$\Delta\omega_d = \frac{v_{\text{atom}}}{c}\omega_L \quad (\text{A.6})$$

for small atomic velocities relative to c . For an atomic velocity $v_{\text{atom}} = v_r$, the Doppler shift is simply $2\omega_r$. Finally, if one wishes to create a standing wave that is moving with respect to the lab frame, the two traveling-wave components must have a frequency difference determined by the relation

$$v_{\text{sw}} = \frac{\Delta\omega_{\text{sw}} \lambda}{2\pi} \quad (\text{A.7})$$

because $\Delta\omega_{\text{sw}}/2\pi$ is the beat frequency of the two waves, and $\lambda/2$ is the spatial periodicity of the standing wave. For a standing wave velocity of v_r , Eq. (A.7) gives $\Delta\omega_{\text{sw}} = 4\omega_r$.

A.3 Hyperfine Structure

A.3.1 Energy Level Splittings

The $6^2S_{1/2} \rightarrow 6^2P_{3/2}$ and $6^2S_{1/2} \rightarrow 6^2P_{1/2}$ transitions are the components of a fine-structure doublet, and each of these transitions additionally have hyperfine structure. The fine structure is a result of the coupling between the orbital angular momentum \mathbf{L} of the outer electron and its spin angular momentum \mathbf{S} . The total electron angular momentum is then given by

$$\mathbf{J} = \mathbf{L} + \mathbf{S} , \quad (\text{A.8})$$

and the corresponding quantum number \mathbf{J} must lie in the range

$$|L - S| \leq J \leq L + S . \quad (\text{A.9})$$

(Here we use the convention that the magnitude of \mathbf{J} is $\sqrt{J(J+1)}\hbar$, and the eigenvalue of J_z is $m_J\hbar$.) For the ground state in cesium, $L = 0$ and $S = 1/2$, so $J = 1/2$; for the first excited state, $L = 1$, so $J = 1/2$ or $J = 3/2$. The energy of any particular level is shifted according to the value of J , so the $L = 0 \rightarrow L = 1$ (D line) transition is split into two components, the D₁ line ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) and the D₂ line ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$). The meaning of the energy level labels is as follows: the first number is the principal quantum number of the outer electron, the superscript is $2S + 1$, the letter refers to L (i.e. $S \leftrightarrow L = 0$, $P \leftrightarrow L = 1$, etc.), and the subscript gives the value of J .

The hyperfine structure is a result of the coupling of \mathbf{J} with the total nuclear angular momentum \mathbf{I} . The total atomic angular momentum \mathbf{F} is then given by

$$\mathbf{F} = \mathbf{J} + \mathbf{I} . \quad (\text{A.10})$$

As before, the magnitude of \mathbf{F} can take the values

$$|J - I| \leq F \leq J + I . \quad (\text{A.11})$$

For the cesium ground state, $J = 1/2$ and $I = 7/2$, so $F = 3$ or $F = 4$. For the excited state of the D_2 line ($6^2P_{3/2}$), F can take any of the values 2, 3, 4, or 5, and for the D_1 excited state ($6^2P_{1/2}$), F is either 3 or 4. Again, the atomic energy levels are shifted according to the value of F .

Because the fine structure splitting in cesium is large enough to be resolved by many lasers (~ 42 nm), the two D-line components are generally treated separately. The hyperfine splittings, however, are much smaller, and it is useful to have some formalism to describe the energy shifts. The Hamiltonian that describes the hyperfine structure for each of the D-line components is [Corney77; Arimondo77]

$$H_{\text{hfs}} = A_{\text{hfs}} \mathbf{I} \cdot \mathbf{J} + B_{\text{hfs}} \frac{3(\mathbf{I} \cdot \mathbf{J})^2 + \frac{3}{2} \mathbf{I} \cdot \mathbf{J} - I(I+1)J(J+1)}{2I(2I-1)J(2J-1)} , \quad (\text{A.12})$$

which leads to a hyperfine energy shift of

$$\Delta E_{\text{hfs}} = \frac{1}{2} A_{\text{hfs}} K + B_{\text{hfs}} \frac{\frac{3}{2} K(K+1) - 2I(I+1)J(J+1)}{2I(2I-1)2J(2J-1)} , \quad (\text{A.13})$$

where

$$K = F(F+1) - I(I+1) - J(J+1) , \quad (\text{A.14})$$

A_{hfs} is the magnetic dipole constant, and B_{hfs} is the electric quadrupole constant (although the term with B_{hfs} applies only to the excited manifold of the D_2 transition and not to the levels with $J = 1/2$). These constants for the cesium D line are listed in Table A.5. The value for the ground state A_{hfs} constant is the recommended value from Ref. [Arimondo77]. The constants listed for the $6^2P_{3/2}$ manifold were taken from a more recent and precise measurement by Tanner and Wieman [Tanner88a]. The A_{hfs} constant for the $6^2P_{1/2}$ manifold is a weighted average of a frequency-comb measurement ($A_{\text{hfs}} = 291.922(20)$ MHz) [Udem99] and a crossed-beam laser spectroscopy measurement ($A_{\text{hfs}} = 291.89(8)$ MHz) [Rafac97]. The energy shift given by (A.13) is relative to the unshifted value (the ‘‘center of gravity’’) listed in Table A.3. The hyperfine structure of the D_2 line, along with the energy splitting values, is diagrammed in Fig. A.2.

A.3.2 Interaction with Static External Fields

A.3.2.1 Magnetic Fields

Each of the hyperfine (F) energy levels contains $2F + 1$ magnetic sublevels that determine the angular distribution of the electron wave function. In the absence of external magnetic fields, these sublevels are degenerate. However, when an external magnetic field is applied, their degeneracy is broken. The Hamiltonian describing the atomic interaction with the magnetic field is

$$\begin{aligned} H_B &= \frac{\mu_B}{\hbar}(g_S \mathbf{S} + g_L \mathbf{L} + g_I \mathbf{I}) \cdot \mathbf{B} \\ &= \frac{\mu_B}{\hbar}(g_S S_z + g_L L_z + g_I I_z) B_z \quad , \end{aligned} \quad (\text{A.15})$$

if we take the magnetic field to be along the z -direction (i.e., along the atomic quantization axis). In this Hamiltonian, the quantities g_S , g_L , and g_I are respectively the electron spin, electron orbital, and nuclear “ g -factors” that account for various modifications to the corresponding magnetic dipole moments. The values for these factors are listed in Table A.6, with the sign convention of [Arimondo77]. The value for g_S has been measured very precisely, and the value given is the CODATA recommended value. The value for g_L is approximately 1, but to account for the finite nuclear mass, the quoted value is given by

$$g_L = 1 - \frac{m_e}{m_{\text{nuc}}} \quad , \quad (\text{A.16})$$

which is correct to lowest order in m_e/m_{nuc} , where m_e is the electron mass and m_{nuc} is the nuclear mass [Bethe57]. The nuclear factor g_I accounts for the entire complex structure of the nucleus, and so the quoted value is an experimental measurement [Arimondo77].

If the energy shift due to the magnetic field is small compared to the fine-structure splitting, then J is a good quantum number and the interaction Hamiltonian can be written as

$$H_B = \frac{\mu_B}{\hbar}(g_J J_z + g_I I_z) B_z \quad . \quad (\text{A.17})$$

Here, the *Landé factor* g_J is given by [Bethe57]

$$\begin{aligned} g_J &= g_L \frac{J(J+1) - S(S+1) + L(L+1)}{2J(J+1)} + g_S \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \\ &\simeq 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \quad , \end{aligned} \quad (\text{A.18})$$

where the second, approximate expression comes from taking the approximate values $g_S \simeq 2$ and $g_L \simeq 1$. The expression here does not include corrections due to the complicated multielectron structure of cesium [Bethe57] and QED effects [Labzowsky99], so the values of g_J given in Table A.6 are experimental measurements [Arimondo77].

If the energy shift due to the magnetic field is small compared to the hyperfine splittings, then similarly F is a good quantum number, so the interaction Hamiltonian becomes [Kleinpoppen97]

$$H_B = \mu_B g_F F_z B_z , \quad (\text{A.19})$$

where the hyperfine Landé g -factor is given by

$$\begin{aligned} g_F &= g_J \frac{F(F+1) - I(I+1) + J(J+1)}{2F(F+1)} + g_I \frac{F(F+1) + I(I+1) - J(J+1)}{2F(F+1)} \\ &\simeq g_J \frac{F(F+1) - I(I+1) + J(J+1)}{2F(F+1)} . \end{aligned} \quad (\text{A.20})$$

The second, approximate expression here neglects the nuclear term, which is a correction at the level of 0.1%, since g_I is much smaller than g_J .

For weak magnetic fields, the interaction Hamiltonian H_B perturbs the zero-field eigenstates of H_{hfs} . To lowest order, the levels split linearly according to [Corney77]

$$\Delta E_{|F m_F\rangle} = \mu_B g_F m_F B_z . \quad (\text{A.21})$$

The approximate g_F factors computed from Eq. (A.20) and the corresponding splittings between adjacent magnetic sublevels are given in Fig. A.2. The splitting in this regime is called the *anomalous Zeeman effect*.

For strong fields where the appropriate interaction is described by Eq. (A.17), the interaction term dominates the hyperfine energies, so that the hyperfine Hamiltonian perturbs the strong-field eigenstates $|J m_J I m_I\rangle$. The energies are then given to lowest order by [Alexandrov93]

$$\begin{aligned} E_{|J m_J I m_I\rangle} &= A_{\text{hfs}} m_J m_I \\ &+ B_{\text{hfs}} \frac{3(m_J m_I)^2 + \frac{3}{2} m_J m_I - I(I+1)J(J+1)}{2J(2J-1)I(2I-1)} \\ &+ \mu_B (g_J m_J + g_I m_I) B_z . \end{aligned} \quad (\text{A.22})$$

The energy shift in this regime is called the *Paschen-Back effect*.

For intermediate fields, the energy shift is more difficult to calculate, and in general one must numerically diagonalize $H_{\text{hfs}} + H_B$. A notable exception is the *Breit-Rabi formula* [Breit31; Kleinpoppen97; Corney77], which applies to the ground-state manifold of the D transition:

$$E_{|J=1/2 m_J I m_I\rangle} = -\frac{\Delta E_{\text{hfs}}}{2(2I+1)} + g_I \mu_B m B \pm \frac{\Delta E_{\text{hfs}}}{2} \left(1 + \frac{4m x}{2I+1} + x^2 \right)^{1/2} . \quad (\text{A.23})$$

In this formula, $\Delta E_{\text{hfs}} = A_{\text{hfs}}(I + 1/2)$ is the hyperfine splitting, $m = m_I \pm m_J = m_I \pm 1/2$ (where the \pm sign is taken to be the same as in (A.23)), and

$$x = \frac{(g_J - g_I)\mu_B B}{\Delta E_{\text{hfs}}} . \quad (\text{A.24})$$

In order to avoid a sign ambiguity in evaluating (A.23), the more direct formula

$$E_{|J=1/2, m_J, I, m_I\rangle} = \Delta E_{\text{hfs}} \frac{I}{2I+1} \pm \frac{1}{2}(g_J + 2Ig_I)\mu_B B \quad (\text{A.25})$$

can be used for the two states $m = \pm(I + 1/2)$. The Breit-Rabi formula is useful in finding the small-field shift of the “clock transition” between the $m_F = 0$ sublevels of the two hyperfine ground states, which has no first-order Zeeman shift. Using $m = m_F$ for small magnetic fields, we obtain

$$\Delta\omega_{\text{clock}} = \frac{(g_J - g_I)^2 \mu_B^2}{2\hbar \Delta E_{\text{hfs}}} B^2 \quad (\text{A.26})$$

to second order in the field strength.

If the magnetic field is sufficiently strong that the hyperfine Hamiltonian is negligible compared to the interaction Hamiltonian, then the effect is termed the *normal Zeeman effect* for hyperfine structure. For even stronger fields, there are Paschen-Back and normal Zeeman regimes for the fine structure, where states with different J can mix, and the appropriate form of the interaction energy is Eq. (A.15). Yet stronger fields induce other behaviors, such as the quadratic Zeeman effect [Kleinpoppen97], which are beyond the scope of the present discussion.

The level structure of cesium in the presence of a magnetic field is shown in Figs. A.4-A.6 in the weak-field (anomalous Zeeman) regime through the hyperfine Paschen-Back regime.

A.3.2.2 Electric Fields

An analogous effect, the *dc Stark effect*, occurs in the presence of a static external electric field. The interaction Hamiltonian in this case is [Armstrong71; Schmieder71; Schmieder72]

$$H_E = -\frac{1}{2}\alpha_0 E_z^2 - \frac{1}{2}\alpha_2 E_z^2 \frac{3J_z^2 - J(J+1)}{J(2J-1)} , \quad (\text{A.27})$$

where we have taken the electric field to be along the z -direction, α_0 and α_2 are respectively termed the *scalar* and *tensor polarizabilities*, and the second (α_2) term is nonvanishing only for the $J = 3/2$ level. The first term shifts all the sublevels with a given J together, so that the Stark shift for the $J = 1/2$ states is trivial. The only mechanism for breaking the degeneracy of the hyperfine sublevels in (A.27) is the J_z contribution in the tensor term. This interaction splits the sublevels such that sublevels with the same value of $|m_F|$ remain degenerate. An expression

for the hyperfine Stark shift, assuming a weak enough field that the shift is small compared to the hyperfine splittings, is [Armstrong71]

$$\Delta E_{|J I F m_F\rangle} = -\frac{1}{2}\alpha_0 E_z^2 - \frac{1}{2}\alpha_2 E_z^2 \frac{[3m_F^2 - F(F+1)][3X(X-1) - 4F(F+1)J(J+1)]}{(2F+3)(2F+2)F(2F-1)J(2J-1)}, \quad (\text{A.28})$$

where

$$X = F(F+1) + J(J+1) - I(I+1). \quad (\text{A.29})$$

For stronger fields, when the Stark interaction Hamiltonian dominates the hyperfine splittings, the levels split according to the value of $|m_J|$, leading to an electric-field analog to the Paschen-Back effect for magnetic fields.

The static polarizability is also useful in the context of optical traps that are very far off resonance (i.e., several to many nm away from resonance, where the rotating-wave approximation is invalid), since the optical potential is given in terms of the ground-state polarizability as $V = -1/2\alpha_0 E^2$, where E is the amplitude of the optical field. A more accurate expression for the far-off resonant potential arises by replacing the static polarizability with the frequency-dependent polarizability [Miller00]

$$\alpha_0(\omega) = \frac{(\hbar\omega_0)^2 \alpha_0}{(\hbar\omega_0)^2 - (\hbar\omega)^2}, \quad (\text{A.30})$$

where ω_0 is the resonant frequency of the lowest-energy transition (i.e., the D_1 resonance); this approximate expression is valid for light tuned to the red of the D_1 line.

The cesium polarizabilities are tabulated in Table A.6. Notice that the *differences* in the excited state and ground state scalar polarizabilities are given, rather than the excited state polarizabilities, since these are the quantities that were actually measured experimentally. The polarizabilities given here are in SI units, although they are often given in cgs units (units of cm^3) or atomic units (units of a_0^3 , where the Bohr radius a_0 is given in Table A.1). The SI values can be converted to cgs units via $\alpha[\text{cm}^3] = 5.95531 \times 10^{-22} \alpha[\text{Hz}/(\text{V}/\text{cm})^2]$ [Miller00], and subsequently the conversion to atomic units is straightforward.

The level structure of cesium in the presence of an external dc electric field is shown in Fig. A.7 in the weak-field regime through the electric hyperfine Paschen-Back regime.

A.3.3 Reduction of the Dipole Operator

The strength of the interaction between cesium and nearly-resonant optical radiation is characterized by the dipole matrix elements. Specifically, $\langle F m_F | e\mathbf{r} | F' m'_F \rangle$ denotes the matrix element that couples the two hyperfine sublevels $|F m_F\rangle$ and $|F' m'_F\rangle$ (where the primed variables

refer to the excited states and the unprimed variables refer to the ground states). To calculate these matrix elements, it is useful to factor out the angular dependence and write the matrix element as a product of a Clebsch-Gordan coefficient and a reduced matrix element, using the Wigner-Eckart theorem [Brink62]:

$$\langle F m_F | er_q | F' m'_F \rangle = \langle F || er || F' \rangle \langle F m_F | F' 1 m'_F q \rangle . \quad (\text{A.31})$$

Here, q is an index labeling the component of \mathbf{r} in the spherical basis, and the doubled bars indicate that the matrix element is reduced. We can also write (A.31) in terms of a Wigner 3- j symbol as

$$\langle F m_F | er_q | F' m'_F \rangle = \langle F || er || F' \rangle (-1)^{F'-1+m_F} \sqrt{2F+1} \begin{pmatrix} F' & 1 & F \\ m'_F & q & -m_F \end{pmatrix} . \quad (\text{A.32})$$

Notice that the 3- j symbol (or, equivalently, the Clebsch-Gordan coefficient) vanishes unless the sublevels satisfy $m_F = m'_F + q$. This reduced matrix element can be further simplified by factoring out the F and F' dependence into a Wigner 6- j symbol, leaving a further reduced matrix element that depends only on the L , S , and J quantum numbers [Brink62]:

$$\begin{aligned} \langle F || er || F' \rangle &\equiv \langle J I F || er || J' I' F' \rangle \\ &= \langle J || er || J' \rangle (-1)^{F'+J+1+I} \sqrt{(2F'+1)(2J+1)} \begin{Bmatrix} J & J' & 1 \\ F' & F & I \end{Bmatrix} . \end{aligned} \quad (\text{A.33})$$

Again, this new matrix element can be further factored into another 6- j symbol and a reduced matrix element involving only the L quantum number:

$$\begin{aligned} \langle J || er || J' \rangle &\equiv \langle L S J || er || L' S' J' \rangle \\ &= \langle L || er || L' \rangle (-1)^{J'+L+1+S} \sqrt{(2J'+1)(2L+1)} \begin{Bmatrix} L & L' & 1 \\ J' & J & S \end{Bmatrix} . \end{aligned} \quad (\text{A.34})$$

The numerical value of the $\langle J = 1/2 || er || J' = 3/2 \rangle$ (D_2) and the $\langle J = 1/2 || er || J' = 1/2 \rangle$ (D_1) matrix elements are given in Table A.7. These values were calculated from the expression [Loudon83]

$$\frac{1}{\tau} = \frac{\omega^3}{3\pi\epsilon_0\hbar c^3} \frac{2J+1}{2J'+1} |\langle J || er || J' \rangle|^2 . \quad (\text{A.35})$$

Note that all the equations we have presented here assume the normalization convention

$$\sum_{M'} |\langle J M | er | J' M' \rangle|^2 = \sum_{M'q} |\langle J M | er_q | J' M' \rangle|^2 = |\langle J || er || J' \rangle|^2 . \quad (\text{A.36})$$

There is, however, another common convention (used in Ref. [Tanner95]) that is related to the convention used here by $(J || er || J') = \sqrt{2J+1} \langle J || er || J' \rangle$.

The dipole matrix elements for specific $|F m_F\rangle \rightarrow |F' m'_F\rangle$ transitions are listed in Tables A.9-A.20 as multiples of $\langle J || er || J' \rangle$. The tables are separated by the ground-state F number (3 or 4) and the polarization of the transition (where σ^+ -polarized light couples $m_F \rightarrow m'_F = m_F + 1$, π -polarized light couples $m_F \rightarrow m'_F = m_F$, and σ^- -polarized light couples $m_F \rightarrow m'_F = m_F - 1$).

A.4 Resonance Fluorescence

A.4.1 Symmetries of the Dipole Operator

Although the hyperfine structure of cesium is quite complicated, it is possible to take advantage of some symmetries of the dipole operator in order to obtain relatively simple expressions for the photon scattering rates due to resonance fluorescence. In the spirit of treating the D_1 and D_2 lines separately, we will discuss the symmetries in this section implicitly assuming that the light is interacting with only one of the fine-structure components at a time. First, notice that the matrix elements that couple to any single excited state sublevel $|F' m'_F\rangle$ add up to a factor that is independent of the particular sublevel chosen,

$$\sum_q |\langle F' (m'_F + q) | er_q | F' m'_F \rangle|^2 = \frac{2J+1}{2J'+1} |\langle J || er || J' \rangle|^2, \quad (\text{A.37})$$

as can be verified from the dipole matrix element tables. This factor of $(2J+1)/(2J'+1)$ (which is 1 for the D_1 line or 1/2 for the D_2 line) is the same factor that appears in Eq. (A.35), and is a consequence of the normalization convention (A.36). The interpretation of this symmetry is simply that all the excited state sublevels decay at the same rate Γ , and the decaying population “branches” into various ground state sublevels.

Another symmetry arises from summing the matrix elements from a single ground-state sublevel to the levels in a particular F' energy level:

$$\begin{aligned} S_{FF'} &:= \sum_q (2F'+1)(2J+1) \left\{ \begin{matrix} J & J' & 1 \\ F' & F & I \end{matrix} \right\}^2 |\langle F m_F | F' 1 (m_F - q) q \rangle|^2 \\ &= (2F'+1)(2J+1) \left\{ \begin{matrix} J & J' & 1 \\ F' & F & I \end{matrix} \right\}^2. \end{aligned} \quad (\text{A.38})$$

This sum $S_{FF'}$ is independent of the particular ground state sublevel chosen, and also obeys the sum rule

$$\sum_{F'} S_{FF'} = 1. \quad (\text{A.39})$$

The interpretation of this symmetry is that for an *isotropic* pump field (i.e. a pumping field with equal components in all *three* possible polarizations), the coupling to the atom is independent of how the population is distributed among the sublevels. These factors $S_{FF'}$ (which are listed in Table A.8) provide a measure of the relative strength of each of the $F \rightarrow F'$ transitions. In the case where the incident light is isotropic and couples two of the F levels, the atom can be treated as a two-level atom, with an effective dipole moment given by

$$|d_{\text{iso,eff}}(F \rightarrow F')|^2 = \frac{1}{3} S_{FF'} |\langle J || e\mathbf{r} || J' \rangle|^2. \quad (\text{A.40})$$

The factor of $1/3$ in this expression comes from the fact that any given polarization of the field only interacts with one (of three) components of the dipole moment, so that it is appropriate to average over the couplings rather than sum over the couplings as in (A.38).

When the light is detuned far from the atomic resonance ($\Delta \gg \Gamma$), the light interacts with several hyperfine levels. If the detuning is large compared to the excited-state frequency splittings, then the appropriate dipole strength comes from choosing any ground state sublevel $|F m_F\rangle$ and summing over its couplings to the excited states. In the case of π -polarized light, the sum is independent of the particular sublevel chosen:

$$\sum_{F'} (2F' + 1)(2J + 1) \left\{ \begin{array}{ccc} J & J' & 1 \\ F' & F & I \end{array} \right\}^2 |\langle F m_F | F' 1 m_F 0 \rangle|^2 = \frac{1}{3} . \quad (\text{A.41})$$

This sum leads to an effective dipole moment for far detuned radiation given by

$$|d_{\text{det,eff}}|^2 = \frac{1}{3} |\langle J || e\mathbf{r} || J' \rangle|^2 . \quad (\text{A.42})$$

The interpretation of this factor is also straightforward. Because the radiation is far detuned, it interacts with the full $J \rightarrow J'$ transition; however, because the light is linearly polarized, it interacts with only one component of the dipole operator. Then, because of spherical symmetry, $|\hat{d}|^2 \equiv |e\hat{r}|^2 = e^2(|\hat{x}|^2 + |\hat{y}|^2 + |\hat{z}|^2) = 3e^2|\hat{z}|^2$. Note that this factor of $1/3$ also appears for σ^\pm light, but only when the sublevels are uniformly populated (which, of course, is not the equilibrium configuration for these polarizations). The effective dipole moments for this case and the case of isotropic pumping are given in Table A.7.

A.4.2 Resonance Fluorescence in a Two-Level Atom

In these two cases, where we have an effective dipole moment, the atoms behave like simple two-level atoms. A two-level atom interacting with a monochromatic field is described by the *optical Bloch equations* [Loudon83],

$$\begin{aligned} \dot{\rho}_{gg} &= \frac{i\Omega}{2}(\tilde{\rho}_{ge} - \tilde{\rho}_{eg}) + \Gamma\rho_{ee} \\ \dot{\rho}_{ee} &= -\frac{i\Omega}{2}(\tilde{\rho}_{ge} - \tilde{\rho}_{eg}) - \Gamma\rho_{ee} \\ \dot{\tilde{\rho}}_{ge} &= -(\gamma + i\Delta)\tilde{\rho}_{ge} - \frac{i\Omega}{2}(\rho_{ee} - \rho_{gg}) , \end{aligned} \quad (\text{A.43})$$

where the ρ_{ij} are the matrix elements of the density operator $\rho := |\psi\rangle\langle\psi|$, $\Omega := -\mathbf{d} \cdot \mathbf{E}_0/\hbar$ is the resonant Rabi frequency, \mathbf{d} is the dipole operator, \mathbf{E}_0 is the electric field amplitude ($\mathbf{E} = \mathbf{E}_0 \cos \omega_L t$), $\Delta := \omega_L - \omega_0$ is the detuning of the laser field from the atomic resonance, $\Gamma = 1/\tau$ is the natural decay rate of the excited state, $\gamma := \Gamma/2 + \gamma_c$ is the “transverse” decay rate (where γ_c is a phenomenological decay rate that models collisions), $\tilde{\rho}_{ge} := \rho_{ge} \exp(-i\Delta t)$ is a

“slowly varying coherence,” and $\tilde{\rho}_{ge} = \tilde{\rho}_{eg}^*$. In writing down these equations, we have made the rotating-wave approximation and used a master-equation approach to model spontaneous emission. Additionally, we have ignored any effects due to the motion of the atom and decays or couplings to other auxiliary states. In the case of purely radiative damping ($\gamma = \Gamma/2$), the excited state population settles to the steady state solution

$$\rho_{ee}(t \rightarrow \infty) = \frac{(\Omega/\Gamma)^2}{1 + 4(\Delta/\Gamma)^2 + 2(\Omega/\Gamma)^2} . \quad (\text{A.44})$$

The (steady state) total photon scattering rate (integrated over all directions and frequencies) is then given by $\Gamma\rho_{ee}(t \rightarrow \infty)$:

$$R_{\text{sc}} = \left(\frac{\Gamma}{2}\right) \frac{(I/I_{\text{sat}})}{1 + 4(\Delta/\Gamma)^2 + (I/I_{\text{sat}})} . \quad (\text{A.45})$$

In writing down this expression, we have defined the *saturation intensity* I_{sat} such that

$$\frac{I}{I_{\text{sat}}} = 2 \left(\frac{\Omega}{\Gamma}\right)^2 , \quad (\text{A.46})$$

which gives (with $I = (1/2)c\epsilon_0 E_0^2$)

$$I_{\text{sat}} = \frac{c\epsilon_0\Gamma^2\hbar^2}{4|\hat{\epsilon} \cdot \mathbf{d}|^2} , \quad (\text{A.47})$$

where $\hat{\epsilon}$ is the unit polarization vector of the light field, and \mathbf{d} is the atomic dipole moment. With I_{sat} defined in this way, the on-resonance scattering cross section σ , which is proportional to $R_{\text{sc}}(\Delta = 0)/I$, drops to 1/2 of its weakly pumped value σ_0 when $I = I_{\text{sat}}$. Additionally, the saturation intensity depends on the polarization of the pumping light as well as the atomic alignment, although the smallest saturation intensity ($I_{\text{sat}}(m_F = \pm 4 \rightarrow m'_F = \pm 5)$, discussed below) is often quoted as a representative value. Some saturation intensities corresponding to the discussions in Section A.4.1 are given in Table A.7. A more detailed discussion of the resonance fluorescence from a two-level atom, including the spectral distribution of the emitted radiation, can be found in Ref. [Loudon83].

A.4.3 Optical Pumping

If none of the special situations in Section A.4.1 applies to the fluorescence problem of interest, then the effects of optical pumping must be accounted for. A discussion of the effects of optical pumping in an atomic vapor on the saturation intensity using a rate-equation approach can be found in Ref. [Sagle96]. Here, however, we will carry out an analysis based on the generalization of the optical Bloch equations (A.43) to the degenerate level structure of cesium. The appropriate master equation for the density matrix of a $F_g \rightarrow F_e$ hyperfine transition is

[Steck98; Kennedy94; Cohen-Tannoudji77; Gao93]

$$\begin{aligned}
\frac{\partial}{\partial t} \tilde{\rho}_{\alpha m_{\alpha}, \beta m_{\beta}} = & -\frac{i}{2} \left[\delta_{\alpha e} \sum_{m_g} \Omega(m_{\alpha}, m_g) \tilde{\rho}_{g m_g, \beta m_{\beta}} - \right. \\
& \delta_{g\beta} \sum_{m_e} \Omega(m_e, m_{\beta}) \tilde{\rho}_{\alpha m_{\alpha}, e m_e} \\
& + \delta_{\alpha g} \sum_{m_e} \Omega^*(m_e, m_{\alpha}) \tilde{\rho}_{e m_e, \beta m_{\beta}} - \\
& \left. \delta_{e\beta} \sum_{m_g} \Omega^*(m_{\beta}, m_g) \tilde{\rho}_{\alpha m_{\alpha}, g m_g} \right] \\
& - \delta_{\alpha e} \delta_{e\beta} \Gamma \tilde{\rho}_{\alpha m_{\alpha}, \beta m_{\beta}} \\
& - \delta_{\alpha e} \delta_{g\beta} \frac{\Gamma}{2} \tilde{\rho}_{\alpha m_{\alpha}, \beta m_{\beta}} \\
& - \delta_{\alpha g} \delta_{e\beta} \frac{\Gamma}{2} \tilde{\rho}_{\alpha m_{\alpha}, \beta m_{\beta}} \\
& + \delta_{\alpha g} \delta_{g\beta} \Gamma \sum_{q=-1}^1 \left[\tilde{\rho}_{e(m_{\alpha}+q), e(m_{\beta}+q)} \right. \\
& \quad \times \langle F_e(m_{\alpha}+q) | F_g 1 m_{\alpha} q \rangle \\
& \quad \times \langle F_e(m_{\beta}+q) | F_g 1 m_{\beta} q \rangle \left. \right] \\
& + i(\delta_{\alpha e} \delta_{g\beta} - \delta_{\alpha g} \delta_{e\beta}) \Delta \tilde{\rho}_{\alpha m_{\alpha}, \beta m_{\beta}}
\end{aligned}
\tag{A.48}$$

(pump field)

(dissipation)

(free evolution)

where

$$\Omega(m_e, m_g) = (-1)^{m_e - m_g} \langle F_e m_e | F_g 1 m_g (m_e - m_g) \rangle \Omega_{-(m_e - m_g)} \tag{A.49}$$

is the Rabi frequency between two magnetic sublevels,

$$\Omega_{-q} = -\frac{2 \langle F_e || e r || F_g \rangle E_{-q}^{(+)}}{\hbar} \tag{A.50}$$

is the overall Rabi frequency with polarization $(-q)$ ($E_{-q}^{(+)}$ is the field amplitude associated with the positive-rotating component, with polarization $(-q)$ in the spherical basis), and δ is the Kronecker delta symbol. This master equation ignores coupling to F levels other than the ground (g) and excited (e) levels; hence, this equation is appropriate for a cycling transition such as $F = 4 \rightarrow F' = 5$. Additionally, this master equation assumes purely radiative damping and, as before, does not describe the motion of the atom.

To calculate the scattering rate from a Zeeman-degenerate atom, it is necessary to solve the master equation for the steady-state populations. Then, the total scattering rate is given by

$$R_{sc} = \Gamma P_e = \Gamma \sum_{m_e} \rho_{e m_e, e m_e} \tag{A.51}$$

where P_e is the total population in the excited state. In addition, by including the branching ratios of the spontaneous decay, it is possible to account for the polarization of the emitted radiation. Defining the scattering rate $R_{\text{sc}, -q}$ for the polarization $(-q)$, we have

$$R_{\text{sc}, -q} = \sum_{m_e m_g} |\langle F_e m_e | F_g 1 m_g q \rangle|^2 \rho_{e m_e, e m_e} , \quad (\text{A.52})$$

where, as before, the only nonzero Clebsch-Gordan coefficients occur for $m_e = m_g + q$. As we have defined it here, $q = \pm 1$ corresponds to σ^\pm -polarized radiation, and $q = 0$ corresponds to π -polarized radiation. The angular distribution for the σ^\pm scattered light is simply the classical radiation pattern for a rotating dipole,

$$f_{\text{sc}}^\pm(\theta, \phi) = \frac{3}{16\pi} (1 + \cos^2 \theta) , \quad (\text{A.53})$$

and the angular distribution for the π -scattered light is the classical radiation pattern for an oscillating dipole,

$$f_{\text{sc}}^0(\theta, \phi) = \frac{3}{8\pi} \sin^2 \theta . \quad (\text{A.54})$$

The net angular pattern will result from the interference of these three distributions.

In general, this master equation is difficult to treat analytically, and even a numerical solution of the time-dependent equations can be time-consuming if a large number of degenerate states are involved. In the following discussions, we will only consider some simple light configurations interacting with the $F = 4 \rightarrow F' = 5$ cycling transition that can be treated analytically. Discussions of Zeeman-degenerate atoms and their spectra can be found in Refs. [Gao93; Gao94a; Gao94b; Polder76; Javanainen92].

A.4.3.1 Circularly (σ^\pm) Polarized Light

The cases where the atom is driven by either σ^+ or σ^- light (i.e. circularly polarized light with the atomic quantization axis aligned with the light propagation direction) are straightforward to analyze. In these cases, the light transfers its angular momentum to the atom, and thus the atomic population is transferred to the state with the largest corresponding angular momentum. In the case of the $F = 4 \rightarrow F' = 5$ cycling transition, a σ^+ driving field will transfer all the atomic population into the $|F = 4, m_F = 4\rangle \rightarrow |F' = 5, m'_F = 5\rangle$ cycling transition, and a σ^- driving field will transfer all the population into the $|F = 4, m_F = -4\rangle \rightarrow |F' = 5, m'_F = -5\rangle$ cycling transition. In both cases, the dipole moment $d_{(m_F=\pm 4 \rightarrow m'_F=\pm 5)}$ is given in Table A.7. Note that these dipole moments are only valid in steady state. If the pumping field is weak, the ‘‘settling time’’ of the atom to its steady state can be long, resulting in a time-dependent effective dipole moment (and saturation intensity). For example, beginning with a uniform sublevel population in the $F = 4$ ground level, the saturation intensity will begin at 2.70 mW/cm² and equilibrate at 1.10 mW/cm² for a circularly polarized pump. Also, if there are any

“remixing” effects such as collisions or magnetic fields not aligned with the axis of quantization, the system may equilibrate to some other configuration.

A.4.3.2 Linearly (π) Polarized Light

If the light is π -polarized (linearly polarized along the quantization axis), the equilibrium population distribution is more complicated. In this case, the atoms tend to accumulate in the sublevels near $m = 0$. Gao [Gao93] has derived analytic expressions for the equilibrium populations of each sublevel and showed that the equilibrium excited-state population is given by Eq. (A.44) if Ω^2 is replaced by

$$g_s(2F_e + 1)|\Omega_0|^2, \quad (\text{A.55})$$

where Ω_0 is the only nonzero component of the Rabi-frequency vector and g_s is a (constant) geometric factor that accounts for the optical pumping. For the cesium $F = 4 \rightarrow F' = 5$ cycling transition, this factor has the value $g_s = 4420/92377 \approx 0.04785$, leading to a steady-state saturation intensity of $I_{\text{sat}} = 1.71 \text{ mW/cm}^2$.

A.4.3.3 One-Dimensional $\sigma^+ - \sigma^-$ Optical Molasses

We now consider the important case of an optical molasses in one dimension formed by one σ^+ and one σ^- field (e.g., by two right-circularly polarized, counterpropagating laser fields). These fields interfere to form a field that is linearly polarized, where the polarization vector traces out a helix in space. Because the light is linearly polarized everywhere, and the steady-state populations are independent of the polarization direction (in the plane orthogonal to the axis of quantization), the analysis of the previous section applies. When we apply the formula (A.45) to calculate the scattering rate, then, we simply use the saturation intensity calculated in the previous section, and use the total intensity (twice the single-beam intensity) for I in the formula.

A.4.3.4 Three-Dimensional Optical Molasses

Finally, we consider an optical molasses in three dimensions, composed of six circularly polarized beams. This optical configuration is found in the commonly used six-beam magneto-optic trap (MOT). However, as we shall see, this optical configuration is quite complicated, and we will only be able to estimate the total rate of fluorescence.

First, we will derive an expression for the electric field and intensity of the light. A typical MOT is formed with two counterpropagating, right-circularly polarized beams along the z -axis and two pairs of counterpropagating, left-circularly polarized beams along the x - and y -

axes. Thus, the net electric field is given by

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= \frac{E_0}{2} e^{-i\omega t} \left[e^{ikz} \left(\frac{\hat{x} - i\hat{y}}{\sqrt{2}} \right) + e^{-ikz} \left(\frac{\hat{x} + i\hat{y}}{\sqrt{2}} \right) \right. \\
&\quad + e^{ikx} \left(\frac{\hat{y} + i\hat{z}}{\sqrt{2}} \right) + e^{-ikx} \left(\frac{\hat{y} - i\hat{z}}{\sqrt{2}} \right) \\
&\quad \left. + e^{iky} \left(\frac{\hat{z} + i\hat{x}}{\sqrt{2}} \right) + e^{-iky} \left(\frac{\hat{z} - i\hat{x}}{\sqrt{2}} \right) \right] + \text{c.c.} \\
&= \sqrt{2} E_0 e^{-i\omega t} \left[(\cos kz - \sin ky) \hat{x} + (\sin kz + \cos kx) \hat{y} + (\cos ky - \sin kx) \hat{z} \right].
\end{aligned} \tag{A.56}$$

Hence, the polarization is linear everywhere, but the orientation of the polarization vector is strongly position-dependent. The corresponding intensity is given by

$$I(\mathbf{r}) = I_0 \left[6 - 4(\cos kz \sin ky + \cos ky \sin kx - \sin kz \cos kx) \right], \tag{A.57}$$

where $I_0 := (1/2)c\epsilon_0 E_0^2$ is the intensity of a single beam. The six beams form an intensity lattice in space, with an average intensity of $6I_0$ and a discrete set of points with zero intensity. Note, however, that the form of this interference pattern is specific to the set of phases chosen here, since there are more than the minimal number of beams needed to determine the lattice pattern.

It is clear that this situation is quite complicated, because an atom moving in this molasses will experience both a changing intensity and polarization direction. The situation becomes even more complicated when the magnetic field gradient from the MOT is taken into account. However, we can estimate the scattering rate if we ignore the magnetic field and assume that the atoms do not remain localized in the lattice, so that they are, on the average, illuminated by all polarizations with intensity $6I_0$. In this case, the scattering rate is given by the two-level atom expression (A.45), with the saturation intensity corresponding to an isotropic pump field ($I_{\text{sat}} = 2.70 \text{ mW/cm}^2$ for the $F = 4 \rightarrow F' = 5$ cycling transition, ignoring the scattering from any light tuned to the $F = 3 \rightarrow F' = 4$ repump transition). Of course, this is almost certainly an overestimate of the effective saturation intensity, since sub-Doppler cooling mechanisms will lead to optical pumping and localization in the light maxima [Townsend95]. These effects can be minimized, for example, by using a very large intensity to operate in the saturated limit, where the scattering rate approaches $\Gamma/2$.

This estimate of the scattering rate is quite useful since it can be used to calculate the number of atoms in an optical molasses from a measurement of the optical scattering rate. For example, if the atoms are imaged by a CCD camera, then the number of atoms N_{atoms} is given by

$$N_{\text{atoms}} = \frac{8\pi \left[1 + 4(\Delta/\Gamma)^2 + (6I_0/I_{\text{sat}}) \right]}{\Gamma(6I_0/I_{\text{sat}}) t_{\text{exp}} \eta_{\text{count}} d\Omega} N_{\text{counts}}, \tag{A.58}$$

where I_0 is the intensity of one of the six beams, N_{counts} is the integrated number of counts recorded on the CCD chip, t_{exp} is the CCD exposure time, η_{count} is the CCD camera efficiency (in counts/photon), and $d\Omega$ is the solid angle of the light collected by the camera. An expression for the solid angle is

$$d\Omega = \frac{\pi}{4} \left(\frac{f}{(f/\#)d_0} \right)^2, \quad (\text{A.59})$$

where f is the focal length of the imaging lens, d_0 is the object distance (from the MOT to the lens aperture), and $f/\#$ is the f -number of the imaging system.

A.5 Data Tables

Table A.1: Fundamental Physical Constants (1998 CODATA recommended values [Mohr00])

Speed of Light	c	$2.997\,924\,58 \times 10^8$ m/s (exact)
Permeability of Vacuum	μ_0	$4\pi \times 10^{-7}$ N/A ² (exact)
Permittivity of Vacuum	ϵ_0	$(\mu_0 c^2)^{-1}$ (exact) $= 8.854\,187\,817 \dots \times 10^{-12}$ F/m
Planck's Constant	h	$6.626\,068\,76(52) \times 10^{-34}$ J·s $4.135\,667\,27(16) \times 10^{-15}$ eV·s
	\hbar	$1.054\,571\,596(82) \times 10^{-34}$ J·s $6.582\,118\,89(26) \times 10^{-16}$ eV·s
Elementary Charge	e	$1.602\,176\,462(63) \times 10^{-19}$ C
Bohr Magneton	μ_B	$9.274\,008\,99(37) \times 10^{-24}$ J/T $h \cdot 1.399\,624\,624(56)$ MHz/G
Atomic Mass Unit	u	$1.660\,538\,73(13) \times 10^{-27}$ kg
Electron Mass	m_e	$5.485\,799\,110(12) \times 10^{-4}$ u $9.109\,381\,88(72) \times 10^{-31}$ kg
Bohr Radius	a_0	$0.529\,177\,208\,3(19) \times 10^{-10}$ m
Boltzmann's Constant	k_B	$1.380\,650\,3(24) \times 10^{-23}$ J/K

Table A.2: Cesium Physical Properties.

Atomic Number	Z	55	
Total Nucleons	$Z + N$	133	
Atomic Mass	m	132.905 451 931(27) u $2.206\,946\,50(17) \times 10^{-25}$ kg	[Bradley99]
Density at 25°C	ρ_m	1.93 g/cm ³	[Lide00]
Melting Point	T_M	28.44 °C	[Lide00]
Boiling Point	T_B	671 °C	[Lide00]
Specific Heat Capacity	c_p	0.242 J/g·K	[Lide00]
Molar Heat Capacity	C_p	32.210 J/mol·K	[Lide00]
Vapor Pressure at 25°C	P_V	1.3×10^{-6} torr	[Nesmeyanov63]
Nuclear Spin	I	7/2	
Ionization Limit	E_I	31 406.467 66(15) cm ⁻¹ 3.893 905(15) eV	[Weber87]

Table A.3: Cesium D₂ ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$) Transition Optical Properties.

Frequency	ω_0	$2\pi \cdot 351.725\,718\,50(11)$ THz	[Udem00]
Wavelength (Vacuum)	λ	852.347 275 82(27) nm	
Wavelength (Air)	λ_{air}	852.118 73 nm	
Wave Number (Vacuum)	$k_L/2\pi$	11 732.307 104 9(37) cm^{-1}	
Lifetime	τ	30.517(57) ns	[Rafac99; Young94]
Decay Rate/ Natural Line Width (FWHM)	Γ	$32.768(62) \times 10^6 \text{ s}^{-1}$ $2\pi \cdot 5.2152(98)$ MHz	
Recoil Velocity	v_r	3.5225 mm/s	
Recoil Energy	ω_r	$2\pi \cdot 2.0663$ kHz	
Doppler shift ($v_{\text{atom}} = v_r$)	$\Delta\omega_d(v_{\text{atom}} = v_r)$	$2\pi \cdot 4.1327$ kHz	
Frequency shift for standing wave moving with $v_{\text{sw}} = v_r$	$\Delta\omega_{\text{sw}}(v_{\text{sw}} = v_r)$	$2\pi \cdot 8.2653$ kHz	

Table A.4: Cesium D₁ ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) Transition Optical Properties.

Frequency	ω_0	$2\pi \cdot 335.116\,048\,807(41)$ THz	[Udem99]
Wavelength (Vacuum)	λ	894.592 959 86(11) nm	
Wavelength (Air)	λ_{air}	894.353 09 nm	
Wave Number (Vacuum)	$k_L/2\pi$	11 178.268 160 7(14) cm^{-1}	
Lifetime	τ	34.855(57) ns	[Rafac99; Young94]
Decay Rate/ Natural Line Width (FWHM)	Γ	$28.690(47) \times 10^6 \text{ s}^{-1}$ $2\pi \cdot 4.5662(75)$ MHz	
Recoil Velocity	v_r	3.3561 mm/s	
Recoil Energy	ω_r	$2\pi \cdot 1.8758$ kHz	
Doppler shift ($v_{\text{atom}} = v_r$)	$\Delta\omega_d(v_{\text{atom}} = v_r)$	$2\pi \cdot 3.7516$ kHz	
Frequency shift for standing wave moving with $v_{\text{sw}} = v_r$	$\Delta\omega_{\text{sw}}(v_{\text{sw}} = v_r)$	$2\pi \cdot 7.5031$ kHz	

Table A.5: Cesium D Transition Hyperfine Structure Constants.

Magnetic Dipole Constant, $6^2S_{1/2}$	$A_{6^2S_{1/2}}$	$h \cdot 2.298\,157\,942\,5$ GHz (exact)	[Arimondo77]
Magnetic Dipole Constant, $6^2P_{1/2}$	$A_{6^2P_{1/2}}$	$h \cdot 291.920(19)$ MHz	[Udem99; Rafac97]
Magnetic Dipole Constant, $6^2P_{3/2}$	$A_{6^2P_{3/2}}$	$h \cdot 50.275(3)$ MHz	[Tanner88a]
Electric Quadrupole Constant, $6^2P_{3/2}$	$B_{6^2P_{3/2}}$	$h \cdot -0.53(2)$ MHz	[Tanner88a]

Table A.6: Cesium D Transition Magnetic and Electric Field Interaction Parameters.

Electron spin g -factor	g_S	2.002 319 304 373 7(80)	[Mohr00]
Electron orbital g -factor	g_L	0.999 995 87	
Fine structure Landé g -factor	$g_J(6^2S_{1/2})$	2.002 540 32(20)	[Arimondo77]
	$g_J(6^2P_{1/2})$	0.665 90(9)	[Arimondo77]
	$g_J(6^2P_{3/2})$	1.3340(3)	[Arimondo77]
Nuclear g -factor	g_I	-0.000 398 853 95(52)	[Arimondo77]
Clock transition Zeeman shift	$\Delta\omega_{\text{clock}}/B^2$	$2\pi \cdot 427.45 \text{ Hz/G}^2$	
Ground-state polarizability	$\alpha_0(6^2S_{1/2})$	$h \cdot 0.1001(20) \text{ Hz/(V/cm)}^2$	[Miller00]
D ₁ scalar polarizability	$\alpha_0(6^2P_{1/2}) - \alpha_0(6^2S_{1/2})$	$h \cdot 0.2415(24) \text{ Hz/(V/cm)}^2$	[Hunter88]
D ₂ scalar polarizability	$\alpha_0(6^2P_{3/2}) - \alpha_0(6^2S_{1/2})$	$h \cdot 0.308 60(60) \text{ Hz/(V/cm)}^2$	[Tanner88b]
D ₂ tensor polarizability	$\alpha_2(6^2P_{3/2})$	$h \cdot -0.065 29(37) \text{ Hz/(V/cm)}^2$	[Tanner88b]

Table A.7: Cesium Dipole Matrix Elements and Saturation Intensities.

D ₂ (6 ² S _{1/2} → 6 ² P _{3/2}) Transition Dipole Matrix Element	$\langle J = 1/2 e r J' = 3/2 \rangle$	4.4754(59) ea_0 3.7944(50) × 10 ⁻²⁹ C.m
Effective Dipole Moment and Saturation Intensity ($F = 4 \rightarrow F' = 5$) (isotropic light polarization)	$d_{\text{iso,eff}}(F = 4 \rightarrow F' = 5)$	2.0199(27) ea_0 1.7126(23) × 10 ⁻²⁹ C.m
	$I_{\text{sat(iso,eff)}}(F = 4 \rightarrow F' = 5)$	2.7020(36) mW/cm ²
Effective Far-Detuned Dipole Moment and Saturation Intensity (D ₂ line, π -polarized light)	$d_{\text{det,eff,D}_2}$	2.5839(34) ea_0 2.1907(29) × 10 ⁻²⁹ C.m
	$I_{\text{sat(det,eff,D}_2)}$	1.6512(22) mW/cm ²
Dipole Moment and Saturation Intensity, $ F = 4, m_F = \pm 4\rangle \rightarrow F' = 5, m'_F = \pm 5\rangle$ cycling transition (σ^\pm -polarized light)	$d_{(m_F = \pm 4 \rightarrow m'_F = \pm 5)}$	3.1646(42) ea_0 2.6831(36) × 10 ⁻²⁹ C.m
	$I_{\text{sat}(m_F = \pm 4 \rightarrow m'_F = \pm 5)}$	1.1008(15) mW/cm ²
D ₁ (6 ² S _{1/2} → 6 ² P _{1/2}) Transition Dipole Matrix Element	$\langle J = 1/2 e r J' = 1/2 \rangle$	3.1840(37) ea_0 2.6995(31) × 10 ⁻²⁹ C.m
Effective Far-Detuned Dipole Moment and Saturation Intensity (D ₁ line, π -polarized light)	$d_{\text{det,eff,D}_1}$	1.8383(21) ea_0 1.5586(18) × 10 ⁻²⁹ C.m
	$I_{\text{sat(det,eff,D}_1)}$	2.5008(29) mW/cm ²

Table A.8: Cesium Relative Hyperfine Transition Strength Factors $S_{FF'}$ (from Eq. (A.38)).

D ₂ (6 ² S _{1/2} → 6 ² P _{3/2}) transition	S_{45}	11/18	S_{34}	15/56
	S_{44}	7/24	S_{33}	3/8
	S_{43}	7/72	S_{32}	5/14
D ₁ (6 ² S _{1/2} → 6 ² P _{1/2}) transition	S_{44}	5/12	S_{34}	3/4
	S_{43}	7/12	S_{33}	1/4

Table A.9: Cesium D₂ ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$) Dipole Matrix Elements for σ^+ transitions ($F = 4, m_F \rightarrow F', m'_F = m_F + 1$), expressed as multiples of $\langle J = 1/2 || er || J' = 3/2 \rangle$.

$m_F =$	-4	-3	-2	-1	0	1	2	3	4
$F' = 5$	$\sqrt{\frac{1}{90}}$	$\sqrt{\frac{1}{30}}$	$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{1}{9}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{7}{30}}$	$\sqrt{\frac{14}{45}}$	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{1}{2}}$
$F' = 4$	$\sqrt{\frac{7}{120}}$	$\sqrt{\frac{49}{480}}$	$\sqrt{\frac{21}{160}}$	$\sqrt{\frac{7}{48}}$	$\sqrt{\frac{7}{48}}$	$\sqrt{\frac{21}{160}}$	$\sqrt{\frac{49}{480}}$	$\sqrt{\frac{7}{120}}$	
$F' = 3$	$\sqrt{\frac{7}{72}}$	$\sqrt{\frac{7}{96}}$	$\sqrt{\frac{5}{96}}$	$\sqrt{\frac{5}{144}}$	$\sqrt{\frac{1}{48}}$	$\sqrt{\frac{1}{96}}$	$\sqrt{\frac{1}{288}}$		

Table A.10: Cesium D₂ ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$) Dipole Matrix Elements for π transitions ($F = 4, m_F \rightarrow F', m'_F = m_F$), expressed as multiples of $\langle J = 1/2 || er || J' = 3/2 \rangle$.

$m_F =$	-4	-3	-2	-1	0	1	2	3	4
$F' = 5$	$-\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{8}{45}}$	$-\sqrt{\frac{7}{30}}$	$-\sqrt{\frac{4}{15}}$	$-\sqrt{\frac{5}{18}}$	$-\sqrt{\frac{4}{15}}$	$-\sqrt{\frac{7}{30}}$	$-\sqrt{\frac{8}{45}}$	$-\sqrt{\frac{1}{10}}$
$F' = 4$	$-\sqrt{\frac{7}{30}}$	$-\sqrt{\frac{21}{160}}$	$-\sqrt{\frac{7}{120}}$	$-\sqrt{\frac{7}{480}}$	0	$\sqrt{\frac{7}{480}}$	$\sqrt{\frac{7}{120}}$	$\sqrt{\frac{21}{160}}$	$\sqrt{\frac{7}{30}}$
$F' = 3$		$\sqrt{\frac{7}{288}}$	$\sqrt{\frac{1}{24}}$	$\sqrt{\frac{5}{96}}$	$\sqrt{\frac{1}{18}}$	$\sqrt{\frac{5}{96}}$	$\sqrt{\frac{1}{24}}$	$\sqrt{\frac{7}{288}}$	

Table A.11: Cesium D₂ ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$) Dipole Matrix Elements for σ^- transitions ($F = 4, m_F \rightarrow F', m'_F = m_F - 1$), expressed as multiples of $\langle J = 1/2 || er || J' = 3/2 \rangle$.

$m_F =$	-4	-3	-2	-1	0	1	2	3	4
$F' = 5$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{14}{45}}$	$\sqrt{\frac{7}{30}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{9}}$	$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{1}{30}}$	$\sqrt{\frac{1}{90}}$
$F' = 4$		$-\sqrt{\frac{7}{120}}$	$-\sqrt{\frac{49}{480}}$	$-\sqrt{\frac{21}{160}}$	$-\sqrt{\frac{7}{48}}$	$-\sqrt{\frac{7}{48}}$	$-\sqrt{\frac{21}{160}}$	$-\sqrt{\frac{49}{480}}$	$-\sqrt{\frac{7}{120}}$
$F' = 3$			$\sqrt{\frac{1}{288}}$	$\sqrt{\frac{1}{96}}$	$\sqrt{\frac{1}{48}}$	$\sqrt{\frac{5}{144}}$	$\sqrt{\frac{5}{96}}$	$\sqrt{\frac{7}{96}}$	$\sqrt{\frac{7}{72}}$

Table A.12: Cesium D₂ ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$) Dipole Matrix Elements for σ^+ transitions ($F = 3, m_F \rightarrow F', m'_F = m_F + 1$), expressed as multiples of $\langle J = 1/2 || er || J' = 3/2 \rangle$.

$m_F =$	-3	-2	-1	0	1	2	3
$F' = 4$	$\sqrt{\frac{5}{672}}$	$\sqrt{\frac{5}{224}}$	$\sqrt{\frac{5}{112}}$	$\sqrt{\frac{25}{336}}$	$\sqrt{\frac{25}{224}}$	$\sqrt{\frac{5}{32}}$	$\sqrt{\frac{5}{24}}$
$F' = 3$	$\sqrt{\frac{3}{32}}$	$\sqrt{\frac{5}{32}}$	$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{5}{32}}$	$\sqrt{\frac{3}{32}}$	
$F' = 2$	$\sqrt{\frac{5}{14}}$	$\sqrt{\frac{5}{21}}$	$\sqrt{\frac{1}{7}}$	$\sqrt{\frac{1}{14}}$	$\sqrt{\frac{1}{42}}$		

Table A.13: Cesium D₂ ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$) Dipole Matrix Elements for π transitions ($F = 3, m_F \rightarrow F', m'_F = m_F$), expressed as multiples of $\langle J = 1/2 || er || J' = 3/2 \rangle$.

$m_F =$	-3	-2	-1	0	1	2	3
$F' = 4$	$-\sqrt{\frac{5}{96}}$	$-\sqrt{\frac{5}{56}}$	$-\sqrt{\frac{25}{224}}$	$-\sqrt{\frac{5}{42}}$	$-\sqrt{\frac{25}{224}}$	$-\sqrt{\frac{5}{56}}$	$-\sqrt{\frac{5}{96}}$
$F' = 3$	$-\sqrt{\frac{9}{32}}$	$-\sqrt{\frac{1}{8}}$	$-\sqrt{\frac{1}{32}}$	0	$\sqrt{\frac{1}{32}}$	$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{9}{32}}$
$F' = 2$		$\sqrt{\frac{5}{42}}$	$\sqrt{\frac{4}{21}}$	$\sqrt{\frac{3}{14}}$	$\sqrt{\frac{4}{21}}$	$\sqrt{\frac{5}{42}}$	

Table A.14: Cesium D₂ ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$) Dipole Matrix Elements for σ^- transitions ($F = 3, m_F \rightarrow F', m'_F = m_F - 1$), expressed as multiples of $\langle J = 1/2 || er || J' = 3/2 \rangle$.

$m_F =$	-3	-2	-1	0	1	2	3
$F' = 4$	$\sqrt{\frac{5}{24}}$	$\sqrt{\frac{5}{32}}$	$\sqrt{\frac{25}{224}}$	$\sqrt{\frac{25}{336}}$	$\sqrt{\frac{5}{112}}$	$\sqrt{\frac{5}{224}}$	$\sqrt{\frac{5}{672}}$
$F' = 3$		$-\sqrt{\frac{3}{32}}$	$-\sqrt{\frac{5}{32}}$	$-\sqrt{\frac{3}{16}}$	$-\sqrt{\frac{3}{16}}$	$-\sqrt{\frac{5}{32}}$	$-\sqrt{\frac{3}{32}}$
$F' = 2$			$\sqrt{\frac{1}{42}}$	$\sqrt{\frac{1}{14}}$	$\sqrt{\frac{1}{7}}$	$\sqrt{\frac{5}{21}}$	$\sqrt{\frac{5}{14}}$

Table A.15: Cesium D₁ ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) Dipole Matrix Elements for σ^+ transitions ($F = 4, m_F \rightarrow F', m'_F = m_F + 1$), expressed as multiples of $\langle J = 1/2 || er || J' = 1/2 \rangle$.

$m_F =$	-4	-3	-2	-1	0	1	2	3	4
$F' = 4$	$\sqrt{\frac{1}{12}}$	$\sqrt{\frac{7}{48}}$	$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{5}{24}}$	$\sqrt{\frac{5}{24}}$	$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{7}{48}}$	$\sqrt{\frac{1}{12}}$	
$F' = 3$	$\sqrt{\frac{7}{12}}$	$\sqrt{\frac{7}{16}}$	$\sqrt{\frac{5}{16}}$	$\sqrt{\frac{5}{24}}$	$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{1}{16}}$	$\sqrt{\frac{1}{48}}$		

Table A.16: Cesium D₁ ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) Dipole Matrix Elements for π transitions ($F = 4, m_F \rightarrow F', m'_F = m_F$), expressed as multiples of $\langle J = 1/2 || er || J' = 1/2 \rangle$.

$m_F =$	-4	-3	-2	-1	0	1	2	3	4
$F' = 4$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{3}{16}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{48}}$	0	$\sqrt{\frac{1}{48}}$	$\sqrt{\frac{1}{12}}$	$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{1}{3}}$
$F' = 3$		$\sqrt{\frac{7}{48}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{5}{16}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{5}{16}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{7}{48}}$	

Table A.17: Cesium D₁ ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) Dipole Matrix Elements for σ^- transitions ($F = 4, m_F \rightarrow F', m'_F = m_F - 1$), expressed as multiples of $\langle J = 1/2 || er || J' = 1/2 \rangle$.

$m_F =$	-4	-3	-2	-1	0	1	2	3	4
$F' = 4$		$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{7}{48}}$	$-\sqrt{\frac{3}{16}}$	$-\sqrt{\frac{5}{24}}$	$-\sqrt{\frac{5}{24}}$	$-\sqrt{\frac{3}{16}}$	$-\sqrt{\frac{7}{48}}$	$-\sqrt{\frac{1}{12}}$
$F' = 3$			$\sqrt{\frac{1}{48}}$	$\sqrt{\frac{1}{16}}$	$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{5}{24}}$	$\sqrt{\frac{5}{16}}$	$\sqrt{\frac{7}{16}}$	$\sqrt{\frac{7}{12}}$

Table A.18: Cesium D₁ ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) Dipole Matrix Elements for σ^+ transitions ($F = 3, m_F \rightarrow F', m'_F = m_F + 1$), expressed as multiples of $\langle J = 1/2 || er || J' = 1/2 \rangle$.

$m_F =$	-3	-2	-1	0	1	2	3
$F' = 4$	$-\sqrt{\frac{1}{48}}$	$-\sqrt{\frac{1}{16}}$	$-\sqrt{\frac{1}{8}}$	$-\sqrt{\frac{5}{24}}$	$-\sqrt{\frac{5}{16}}$	$-\sqrt{\frac{7}{16}}$	$-\sqrt{\frac{7}{12}}$
$F' = 3$	$-\sqrt{\frac{1}{16}}$	$-\sqrt{\frac{5}{48}}$	$-\sqrt{\frac{1}{8}}$	$-\sqrt{\frac{1}{8}}$	$-\sqrt{\frac{5}{48}}$	$-\sqrt{\frac{1}{16}}$	

Table A.19: Cesium D₁ ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) Dipole Matrix Elements for π transitions ($F = 3, m_F \rightarrow F', m'_F = m_F$), expressed as multiples of $\langle J = 1/2 || er || J' = 1/2 \rangle$.

$m_F =$	-3	-2	-1	0	1	2	3
$F' = 4$	$\sqrt{\frac{7}{48}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{5}{16}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{5}{16}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{7}{48}}$
$F' = 3$	$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{1}{12}}$	$\sqrt{\frac{1}{48}}$	0	$-\sqrt{\frac{1}{48}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{16}}$

Table A.20: Cesium D₁ ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) Dipole Matrix Elements for σ^- transitions ($F = 3, m_F \rightarrow F', m'_F = m_F - 1$), expressed as multiples of $\langle J = 1/2 || er || J' = 1/2 \rangle$.

$m_F =$	-3	-2	-1	0	1	2	3
$F' = 4$	$-\sqrt{\frac{7}{12}}$	$-\sqrt{\frac{7}{16}}$	$-\sqrt{\frac{5}{16}}$	$-\sqrt{\frac{5}{24}}$	$-\sqrt{\frac{1}{8}}$	$-\sqrt{\frac{1}{16}}$	$-\sqrt{\frac{1}{48}}$
$F' = 3$		$\sqrt{\frac{1}{16}}$	$\sqrt{\frac{5}{48}}$	$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{5}{48}}$	$\sqrt{\frac{1}{16}}$

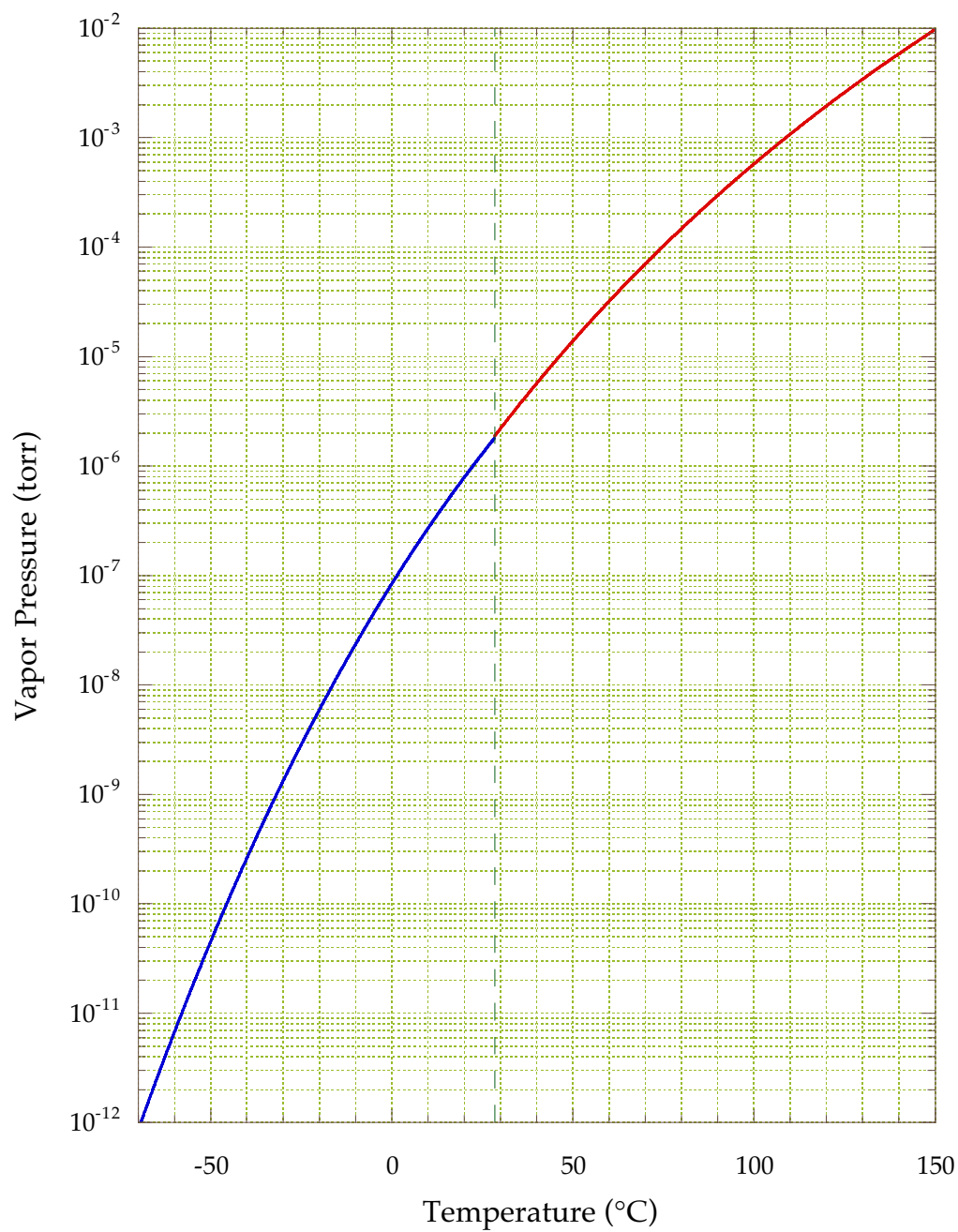


Figure A.1: Vapor pressure of cesium from the model of Eqs. (A.1). The vertical line indicates the melting point.

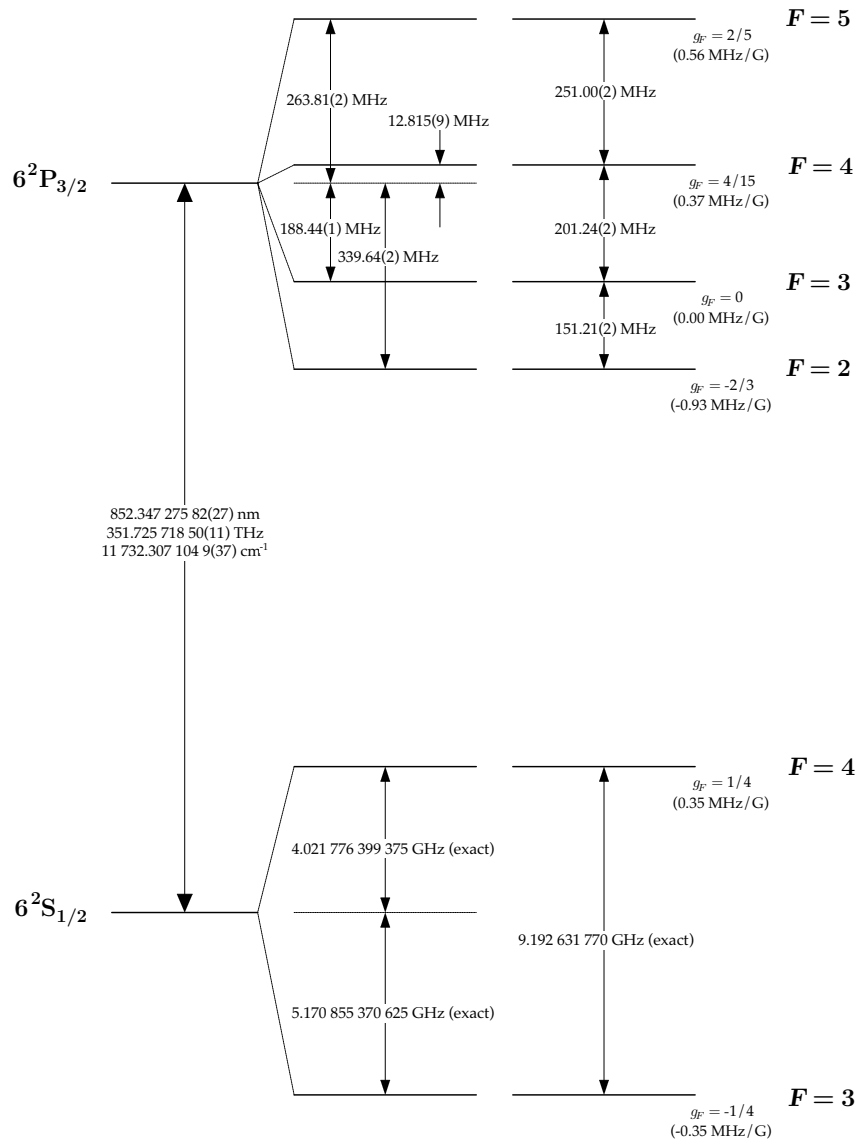


Figure A.2: Cesium D₂ transition hyperfine structure, with frequency splittings between the hyperfine energy levels. The excited-state values are taken from [Tanner88a], and the ground-state values are exact, as a result of the current definition of the second. The approximate Landé g_F -factors for each level are also given, with the corresponding Zeeman splittings between adjacent magnetic sublevels.

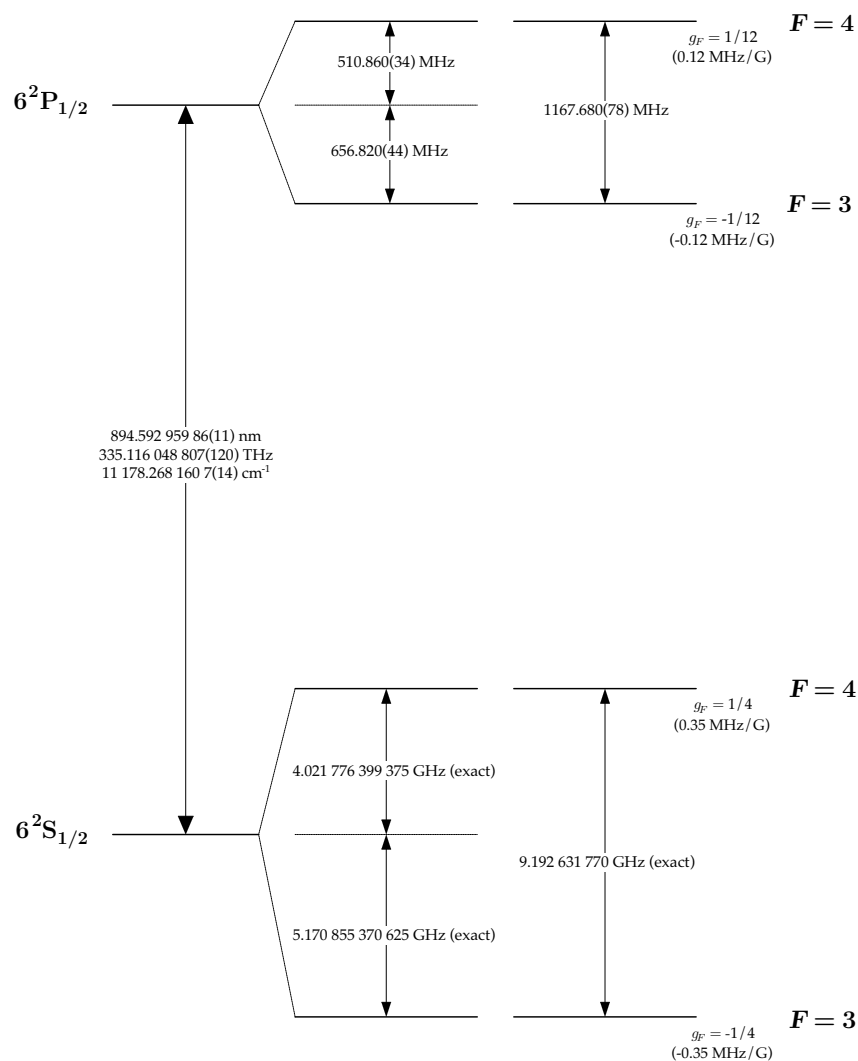


Figure A.3: Cesium D₁ transition hyperfine structure, with frequency splittings between the hyperfine energy levels. The excited-state values are taken from [Udem99; Rafac97], and the ground-state values are exact, as a result of the current definition of the second. The approximate Landé g_F -factors for each level are also given, with the corresponding Zeeman splittings between adjacent magnetic sublevels.

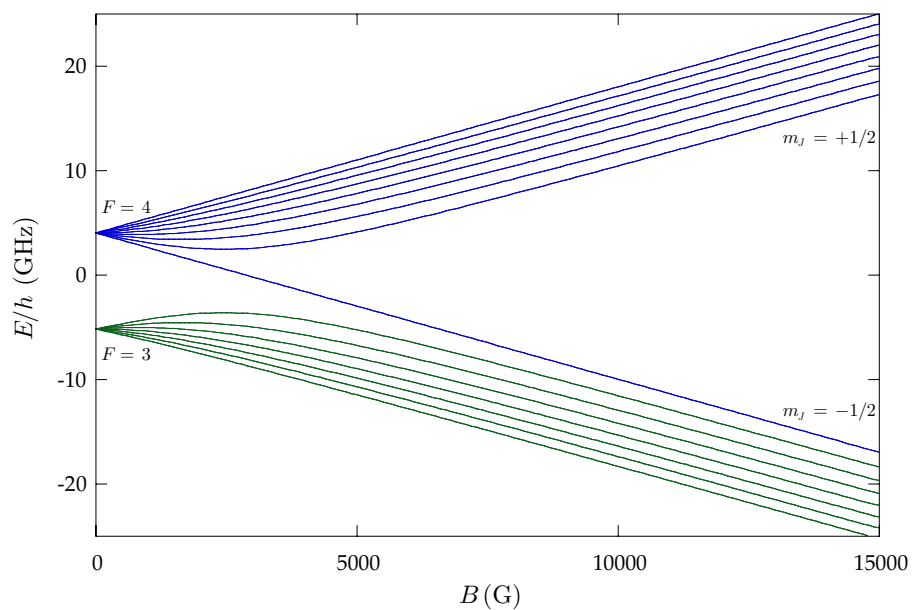


Figure A.4: Cesium $6^2S_{1/2}$ (ground) level hyperfine structure in an external magnetic field. The levels are grouped according to the value of F in the low-field (anomalous Zeeman) regime and m_J in the strong-field (hyperfine Paschen-Back) regime.

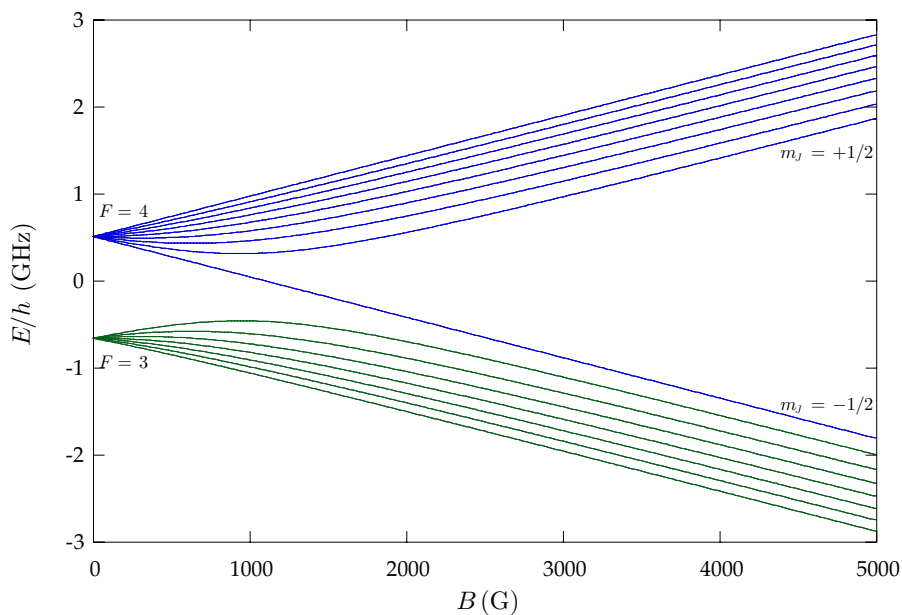


Figure A.5: Cesium $6^2P_{1/2}$ (D_1 excited) level hyperfine structure in an external magnetic field. The levels are grouped according to the value of F in the low-field (anomalous Zeeman) regime and m_J in the strong-field (hyperfine Paschen-Back) regime.

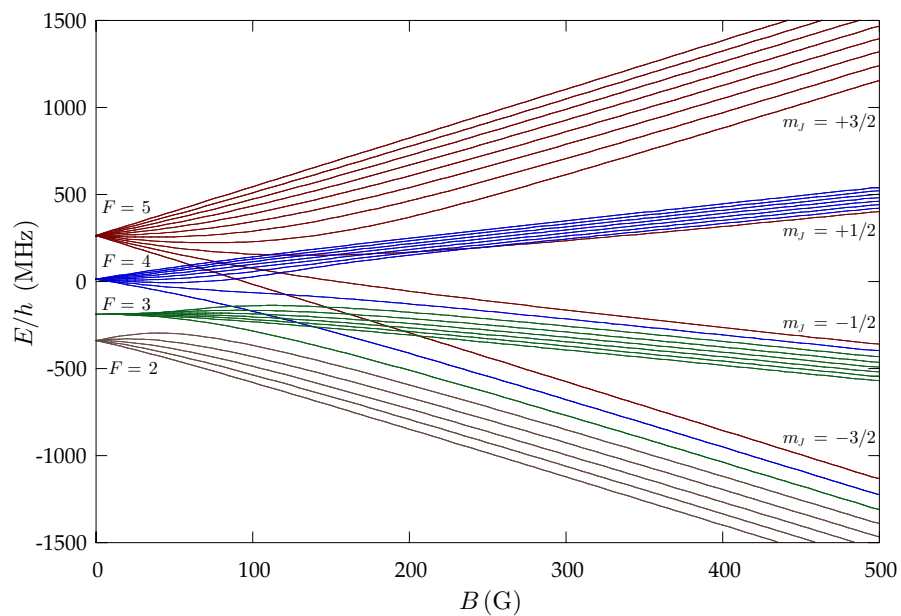


Figure A.6: Cesium $6^2P_{3/2}$ (D_2 excited) level hyperfine structure in an external magnetic field. The levels are grouped according to the value of F in the low-field (anomalous Zeeman) regime and m_J in the strong-field (hyperfine Paschen-Back) regime.

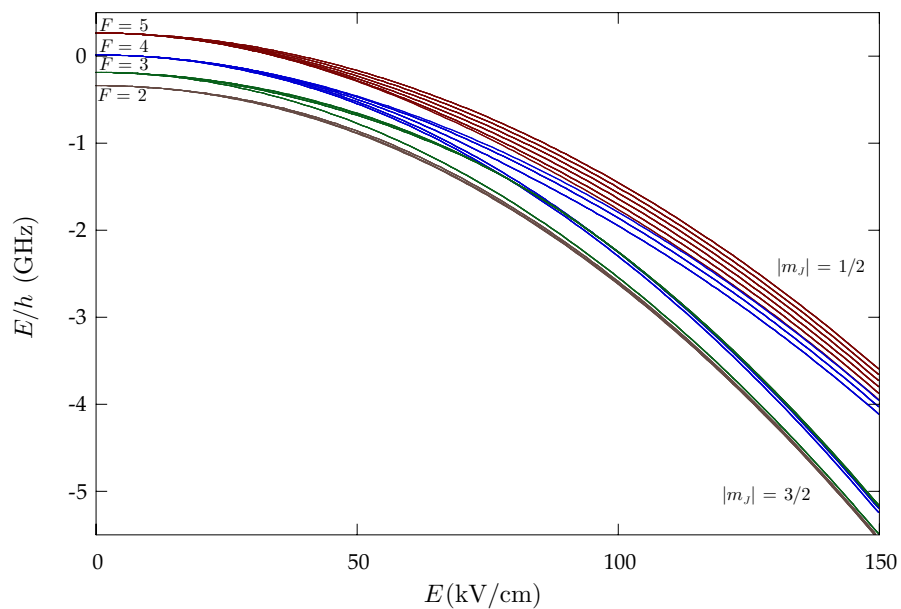


Figure A.7: Cesium $6^2P_{3/2}$ (D_2 excited) level hyperfine structure in a constant, external electric field. The levels are grouped according to the value of F in the low-field (anomalous Zeeman) regime and $|m_J|$ in the strong-field (“electric” hyperfine Paschen-Back) regime. Levels with the same values of F and $|m_F|$ (for a weak field) are degenerate.

