Appendix B

Phase Space Gallery I: Standard Map

In this appendix we take a stroll through the phase space of the standard map as it becomes increasingly chaotic. As we have seen in Chapter 4, the standard map is actually a one-parameter family of maps, given by

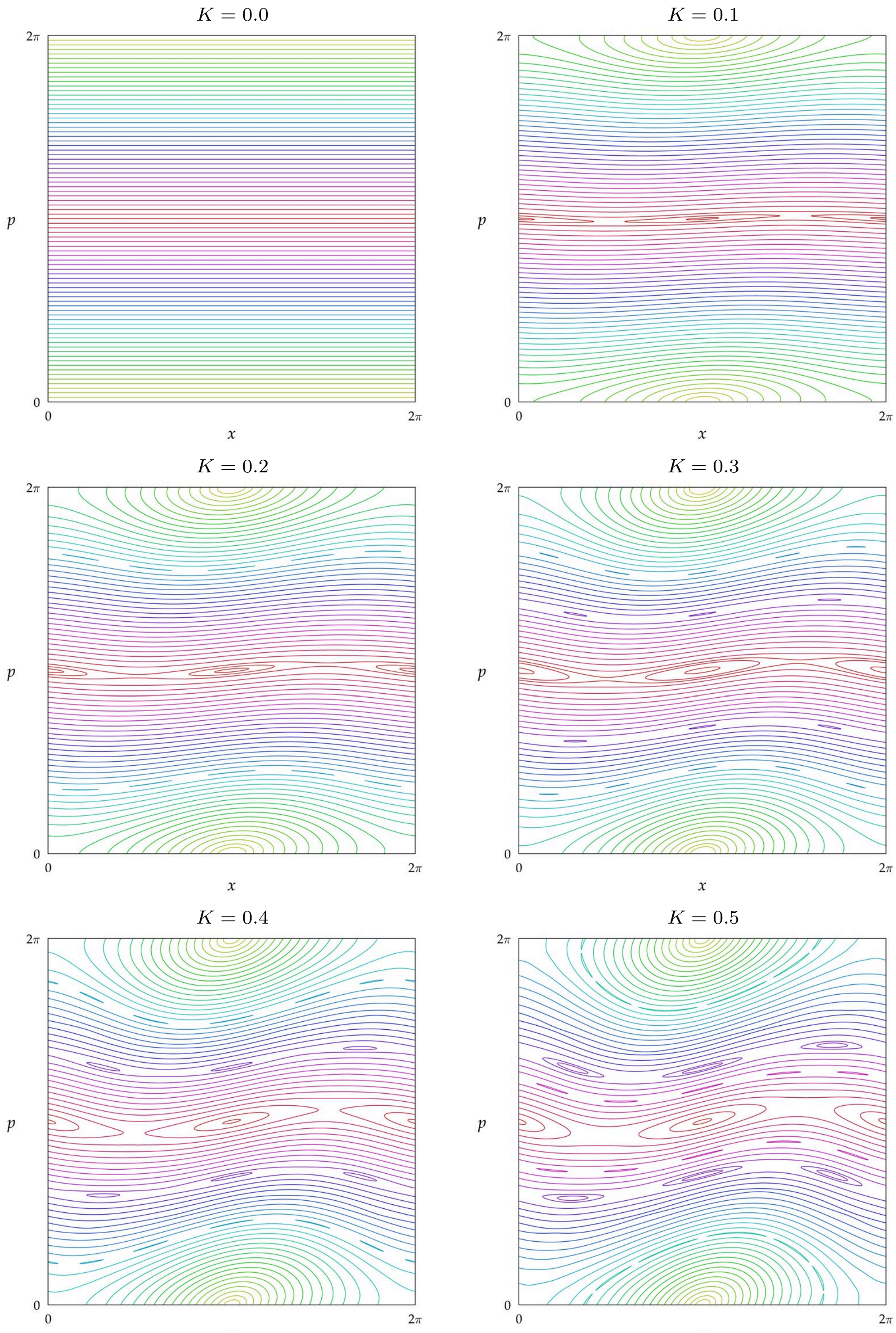
$$\begin{aligned} p_{n+1} &= p_n + K \sin x_n \\ x_{n+1} &= x_n + p_{n+1} \quad . \end{aligned} \tag{B.1}$$

The stochasticity parameter K controls the "degree of chaos" of the dynamics.

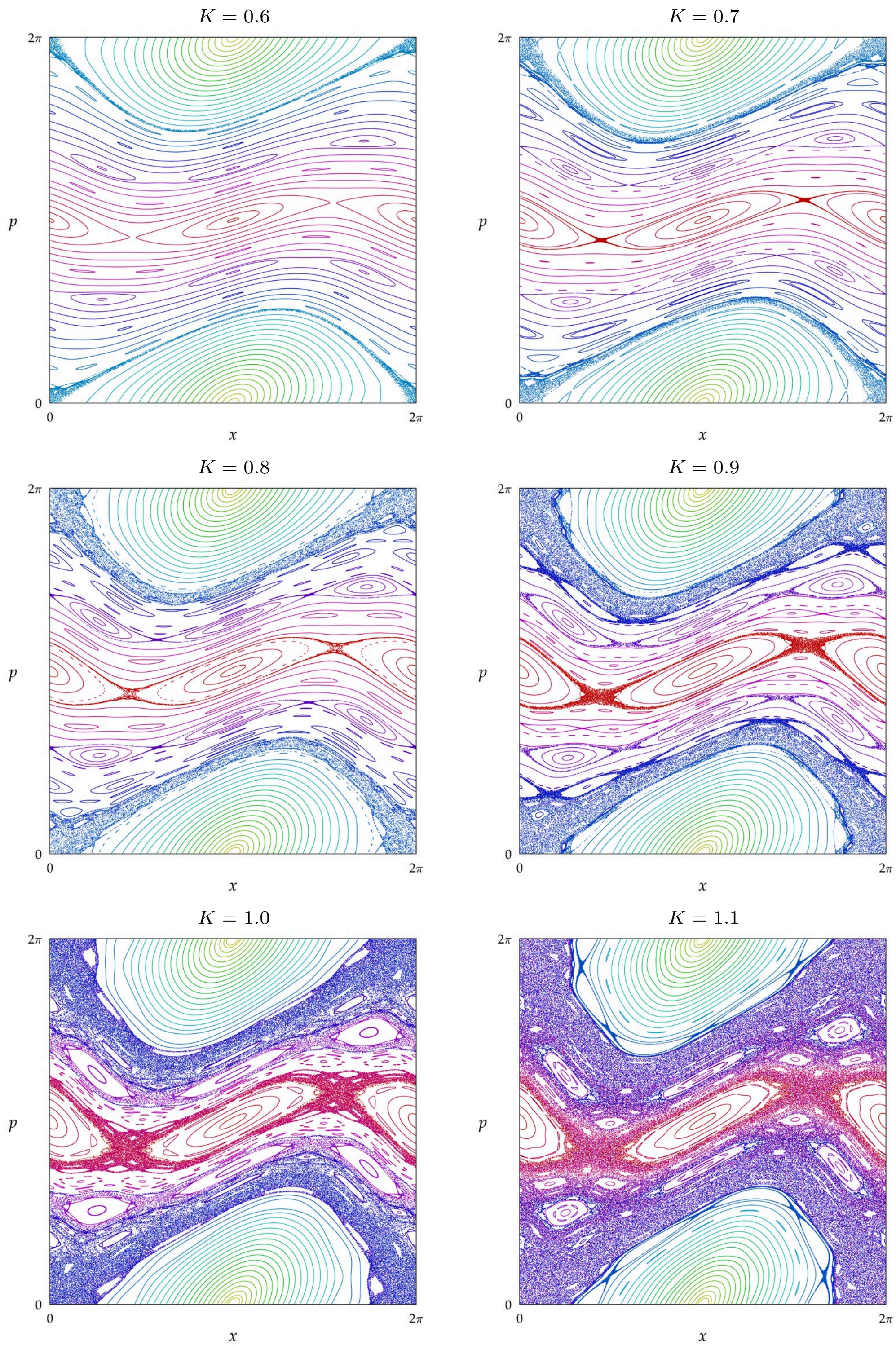
For $K = 0$, the momentum is a constant of the motion, so the phase space is simply the set of contours of constant p. For small K, the map is weakly perturbed. Most of the tori are distorted but still present, with only very small chaotic regions (as expected from the KAM theorem), and several resonances become visible (as expected from the Poincaré–Birkhoff theorem). The phase space structure becomes especially rich when K is near Greene's number (≈ 0.971635) [Greene79], which is the critical value at which the last KAM surface (i.e., invariant torus that spans the phase space, partitioning the chaotic regions into disconnected cells) is destroyed, and the chaotic transport makes a phase transition from local to global diffusion. Beyond this value, much of the stable structure breaks down, leaving mostly the primary resonances, until they become unstable at $K = 4$. The phase space then becomes mostly chaotic, with some islands popping back up now and then (such as the accelerator modes near $K = 2\pi$ and 4π).

In the graphics that follow, the iterates of the standard map are plotted for various stochasticity parameters. The phase plots here show about 40 different trajectories, with around 8000 iterations per trajectory. To retain the symmetry of the phase space, the symmetric image $(2\pi - x, 2\pi - p)$ of each (x, p) point is also plotted. The colors in the diagrams mark different trajectories, serving the dual purposes of heightening the contrast between phase-space structures and making it clear when mixing behavior is present. Because of the 2π-periodicity of the phase space in both x and p, only one unit cell of the phase space is shown.

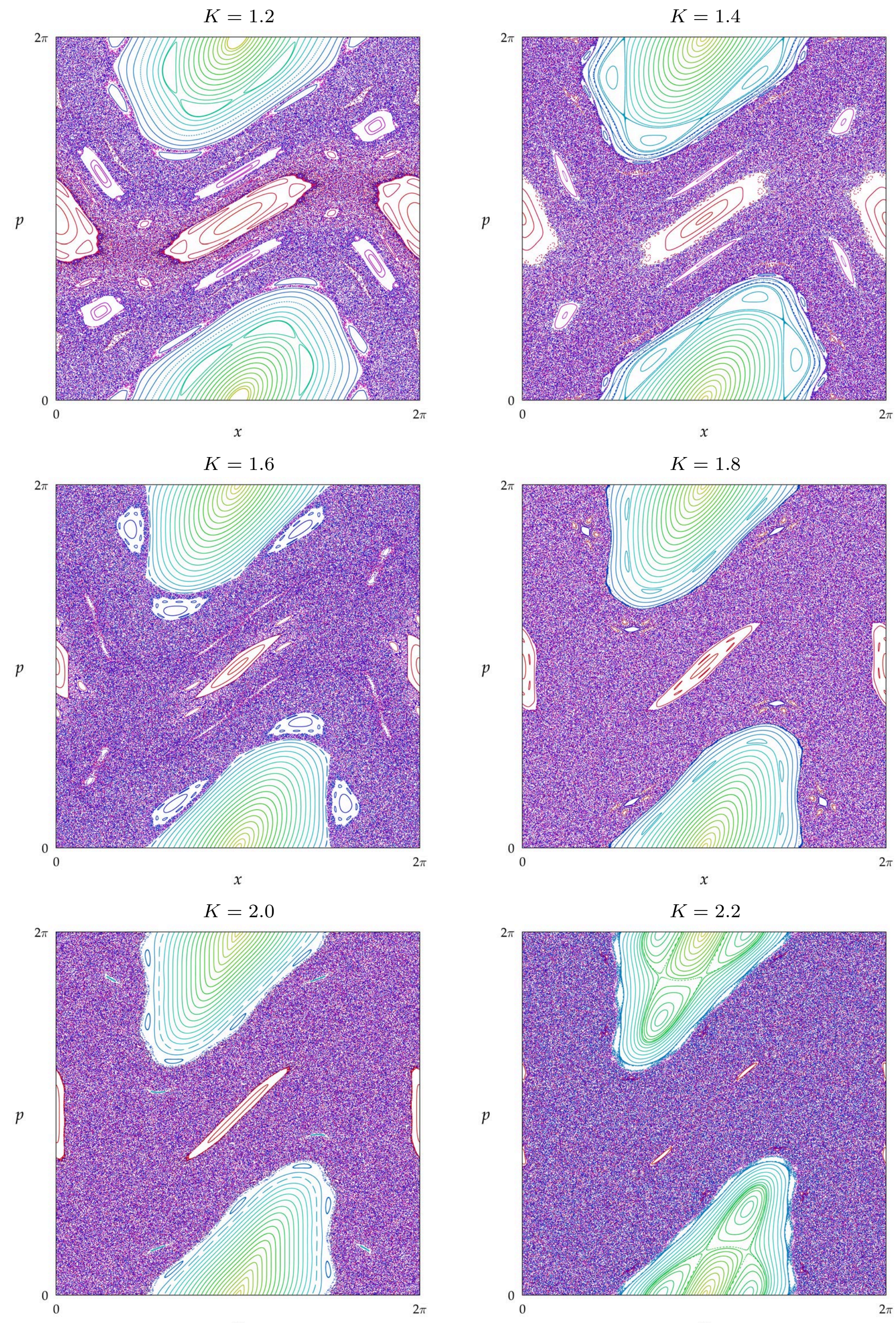
The phase-space plots in this and the next gallery were hand-coded directly in the POSTSCRIPT graphics language, which is a powerful and efficient tool for producing complex graphics. These rather small graphics files can then be printed on standard laser printers, and the printer's processor performs the iteration mapping or numerical integration to determine the locations of the points. To save space in the PDF-formatted version of this document (and for more consistent color and intensity), these phase spaces were rasterized prior to inclusion in this LaTeX document. An earlier example of using PostScript for plotting the phase space of an iterated map can be found in [Peres93].



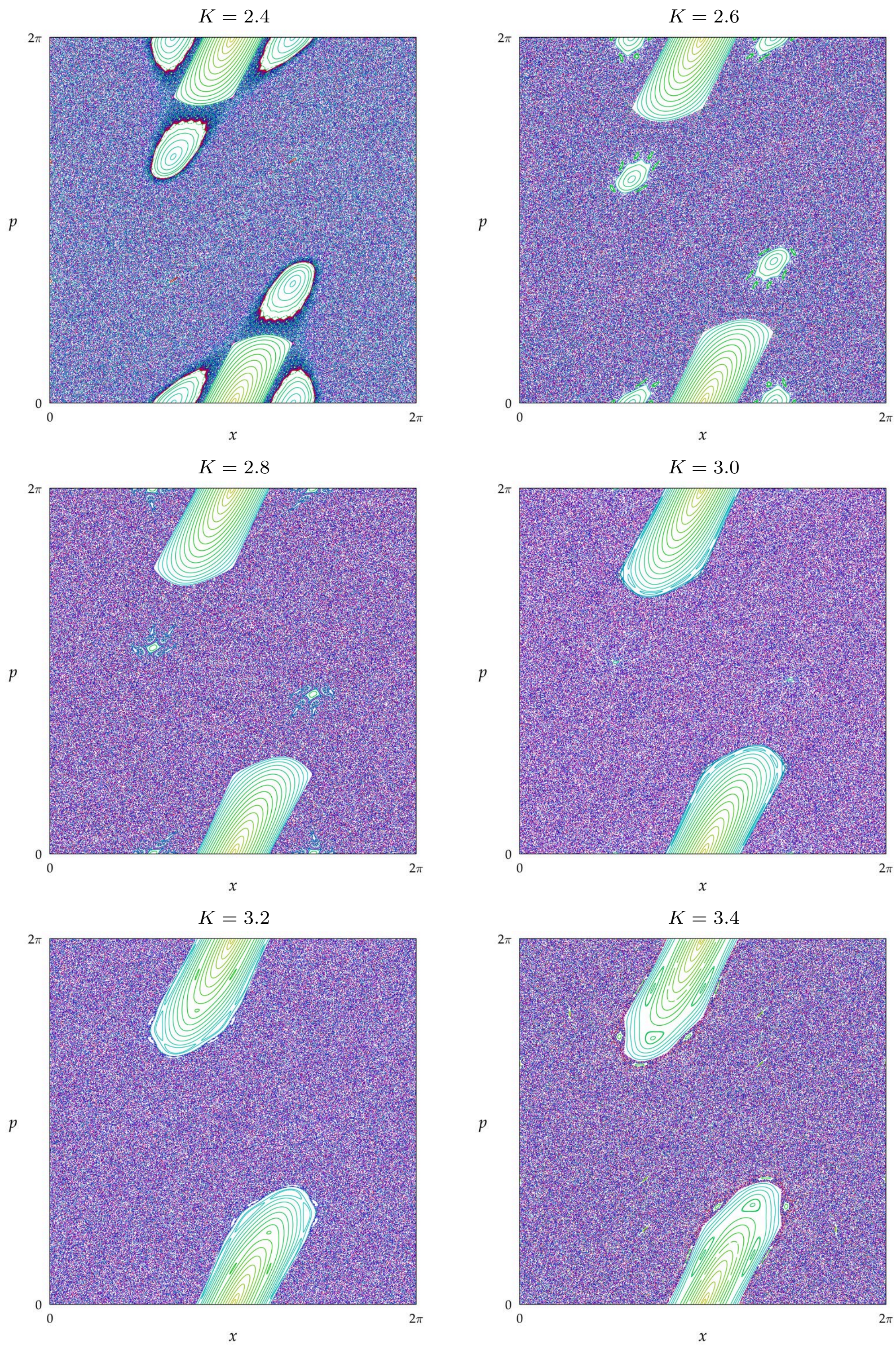
K = 0.0
K = 0.1
K = 0.2
K = 0.3
K = 0.4
K = 0.5
2π
0
p
x

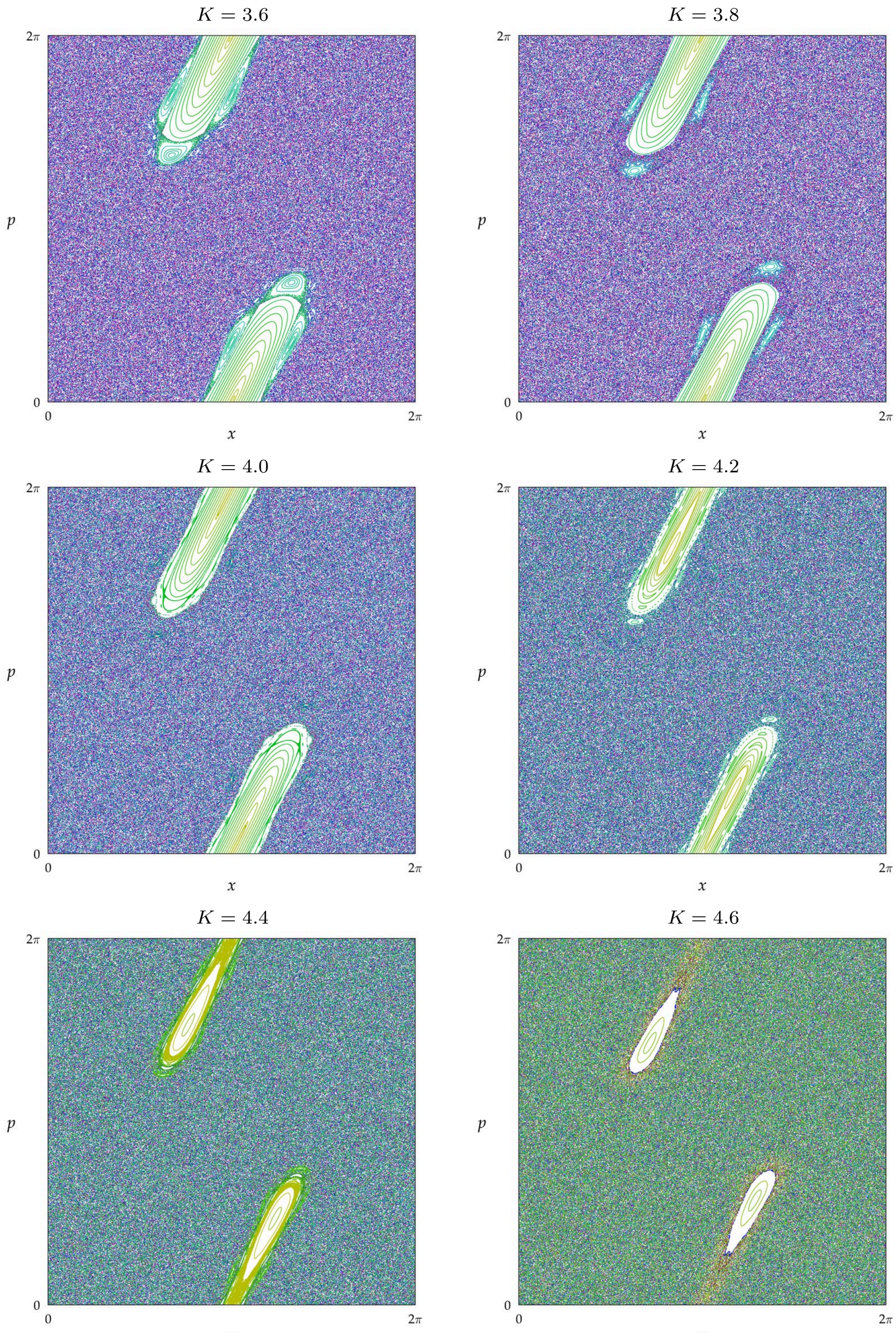


$K = 0.6$
$K = 0.7$
$K = 0.8$
$K = 0.9$
$K = 1.0$
$K = 1.1$
p
x
0
2π

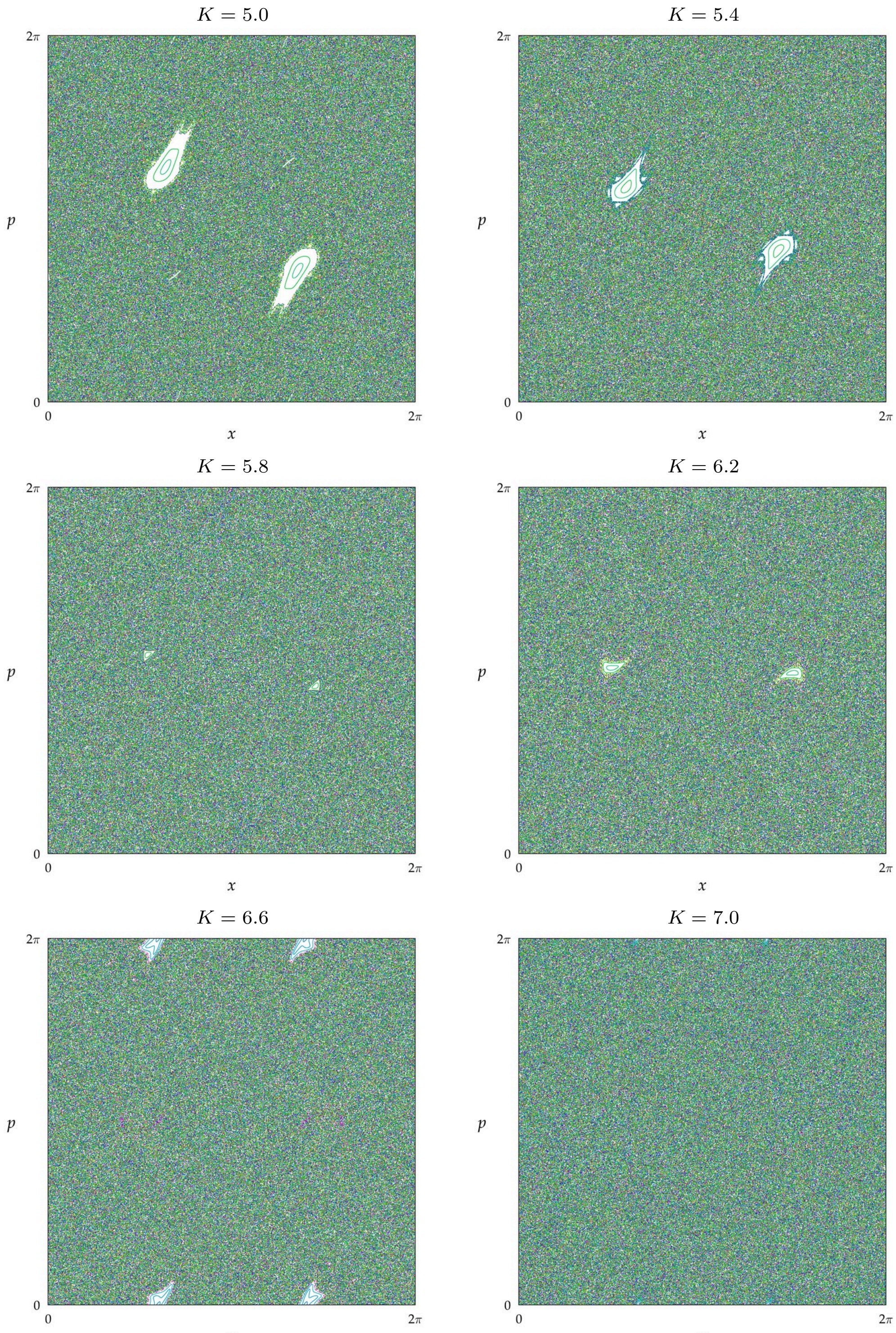


$K = 1.2$
$K = 1.4$
$K = 1.6$
$K = 1.8$
$K = 2.0$
$K = 2.2$
2π
p
0
x

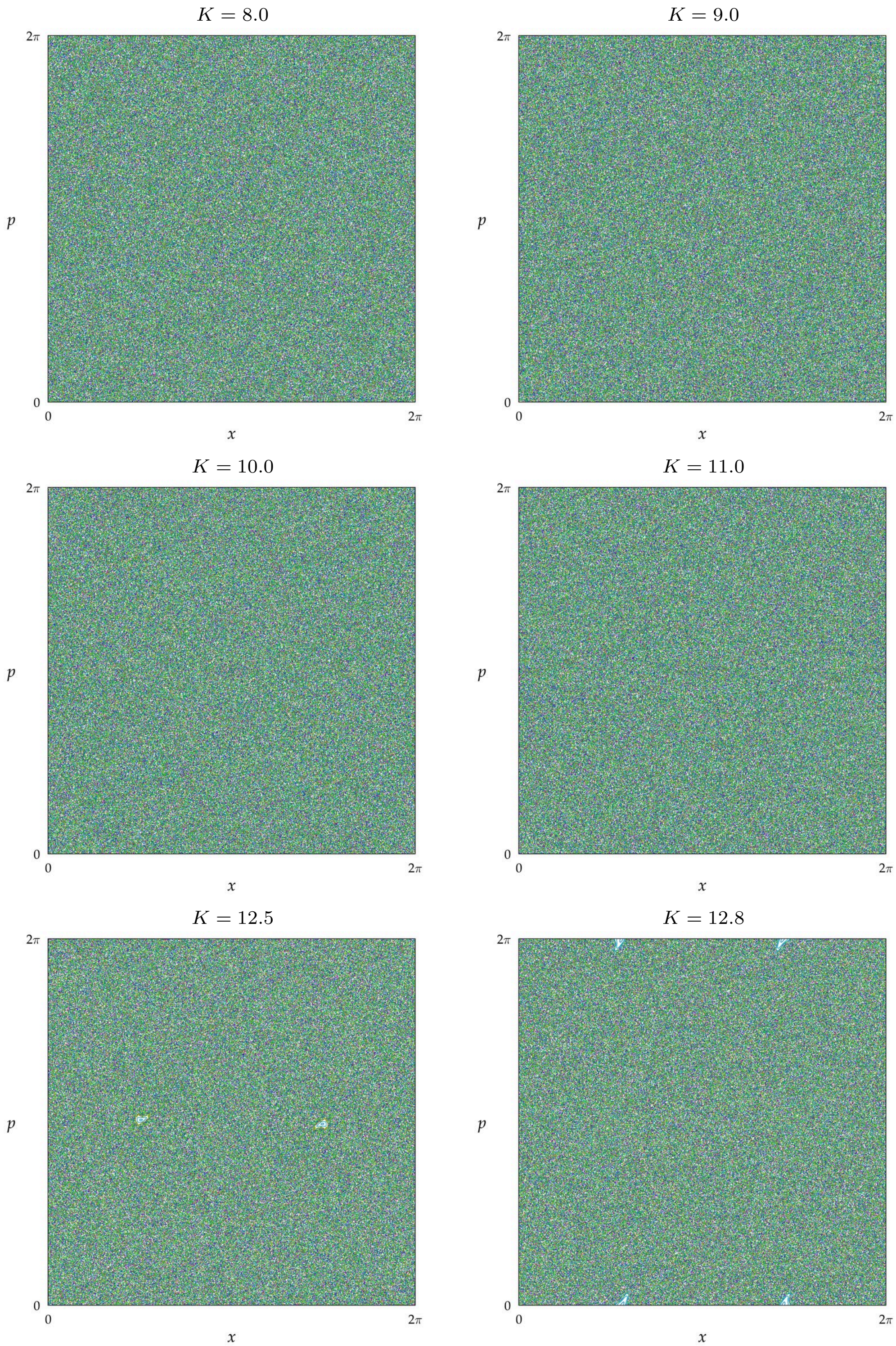

$K = 2.4$
$K = 2.6$
$K = 2.8$
$K = 3.0$
$K = 3.2$
$K = 3.4$
p
x
0
2π



$K = 3.6$
$K = 3.8$
$K = 4.0$
$K = 4.2$
$K = 4.4$
$K = 4.6$
2π
p
0
x



$K = 5.0$
$K = 5.4$
$K = 5.8$
$K = 6.2$
$K = 6.6$
$K = 7.0$
2π
p
0
x


K = 8.0
K = 9.0
K = 10.0
K = 11.0
K = 12.5
K = 12.8
2π
0
p
x