Numerical Quantum Optics Term Project: Feedback Cooling of a Mechanical Oscillator

Mark Rodenberger, Nima Dinyari, and Xiaolu Cheng
Department of Physics and Oregon Center for Optics
1274 University of Oregon
Eugene, Oregon 97403-1274

June, 2007

1 Introduction to the System

The system that our group is modeling is that of an optical cavity formed by two mirrors where one is free to move with mechanical frequency ω_m. Light is pumped into the cavity through a lossy mirror, which forms the other end of the cavity, as in Figure. 1.

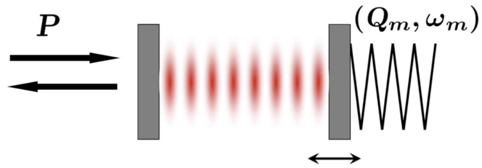

Figure 1: Model of system in consideration. Image taken from [1].

As the radiation pressure builds up in the cavity a force is applied to the mirror which displaces it. Our group's goal is to implement a feedback loop to cool the center of mass motion of the movable mirror. To achieve this we will use information gained about the mean position of the mirror from the measurement record gained via Homodyne Detection [3] of the light emitted.

Once we know the mean position of the mirror we will be able to adjust the pump power incident on the cavity to critically damp the motion of the mirror. We will use parameters in our program of a deformed microsphere, made out of fused silicia, that has a diameter of 30μm which is similar to Young-Shin Park's microspheres that are made in the Wang Lab at the University of Oregon.

2 Equations for program to solve

The Hamiltonian describing our system, in the interaction picture of the free Hamiltonian of the cavity mode, is give by [2]:

$$H = H_m - \hbar g a^\dagger a x + i\hbar E(a - a^\dagger), \tag{1}$$

where:

$$H_m = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{2}$$

is the Hamiltonian for the mechanical motion of the mirror; or $\hbar\omega_m \alpha_m^\dagger \alpha_m$ in the number basis.

The second term in Eq. (1) describes the coupling between the optical cavity mode and the mechanical mode of mirror where $g = \frac{\omega_0}{L}$, ω_0 is the mode frequency and L is the cavity length. The last term describes the coherent driving of the cavity mode where E is related to the pump power, P by $E = \sqrt{\frac{\gamma P}{\hbar\omega_0}}$. Here, $\gamma = 2\pi\delta\nu_{\text{FWHM}}$ is the decay rate of the cavity.

Assuming the detector efficiency is 1, the evolution of the system (cavity + mirror) is described by the Stochastic Master Equation (SME):

$$d\rho_c = -\frac{i}{\hbar}[H, \rho_c]dt + \gamma\mathcal{D}[a]\rho_c dt + \sqrt{\gamma}\mathcal{H}[-ia]\rho_c dW, \tag{3}$$

where ρ_c is the system density matrix, dW is the Wiener increment, satisfying the Ito calculus relation $(dW)^2 = dt$, and the superoperators $\mathcal{D}$ and $\mathcal{H}$ are given by:

$$\mathcal{D}[c]\rho_c = c\rho_c c^\dagger - \frac{1}{2}(c^\dagger c\rho_c + \rho_c c^\dagger c),$$
$$\mathcal{H}[c]\rho_c = c\rho_c + \rho_c c^\dagger - \langle c + c\dagger\rangle\rho_c.$$

We follow the steps outlined in [2] and transform into the displacement picture:

$$\rho_c^{'} = D(-\alpha)\rho_c D^\dagger(-\alpha)$$

where $D(\alpha)$ is the displacement operator, such that $D(\alpha)|0\rangle = |\alpha = \frac{-2E}{\gamma}\rangle$ and $|\alpha\rangle$ is a coherent state of the cavity mode. Note: in what follows we will let $D(-\alpha) \to D$ for brevities sake. We can transform the SME (3) into this picture as follows:

$$d\rho_c^{'} = D(\rho + d\rho)D^\dagger - D(\rho)D^\dagger = D(d\rho_c)D^\dagger.$$

The following relations help in bringing the master equation into the desired form:

$$D^\dagger a D = a + \alpha, \qquad D^\dagger a^\dagger D = a^\dagger + \alpha^*.$$

It also helps to note that α is restricted to real values, so $\alpha = \alpha^*$.

Then sandwiching each of the various terms of the SME with D and $D^\dagger$ we get:

$$D(H_m\rho_c - \rho_c H_m)D^\dagger = H_m D\rho_c D^\dagger - D\rho_c D^\dagger H_m = H_m\rho_c^{'} - \rho_c^{'}H_m \tag{4}$$

for the commutator of the mechanical Hamiltonian. For the commutator with the coupling term we get:

$$\begin{aligned} D(a^\dagger a x\rho_c - \rho_c a^\dagger a x)D^\dagger &= Da^\dagger D^\dagger D a D^\dagger D x\rho_c D^\dagger - D\rho_c D^\dagger D a^\dagger D^\dagger D a D^\dagger x \\ &= (a^\dagger + \alpha^*)(a+\alpha)x\rho_c^{'} - \rho_c^{'}(a^\dagger + \alpha^*)(a+\alpha)x \\ &= [(a^\dagger a + \alpha(a + a^\dagger) + |\alpha|^2)x, \rho_c{'}], \end{aligned} \tag{5}$$

and for the driving term:

$$\begin{aligned} D\left((a - a^\dagger)\rho_c - \rho_c(a - a^\dagger)\right)D^\dagger &= D(a - a^\dagger)D^\dagger D\rho_c D^\dagger - D\rho_c D^\dagger D(a - a^\dagger)D^\dagger \\ &= \left(a + \alpha - (a^\dagger + \alpha^*)\right)\rho_c^{'} - \rho_c^{'}\left(a + \alpha - (a^\dagger + \alpha^*)\right) \\ &= (a - a^\dagger)\rho_c^{'} - \rho_c^{'}(a - a^\dagger). \end{aligned} \tag{6}$$

The Dissipation term becomes:

$$\begin{aligned} D\left(a\rho_c a^\dagger - \tfrac{1}{2}a^\dagger a\rho_c - \tfrac{1}{2}\rho_c a^\dagger a\right)D^\dagger &= (a+\alpha)\rho_c^{'}(a^\dagger + \alpha^*) - \tfrac{1}{2}(a^\dagger + \alpha^*)(a+\alpha)\rho_c^{'} \\ &\quad -\tfrac{1}{2}\rho_c^{'}(a^\dagger + \alpha^*)(a+\alpha) \\ &= a\rho_c^{'}a^\dagger + \alpha(\rho_c^{'}a^\dagger + a\rho_c^{'}) + |\alpha|^2\rho_c^{'} - \tfrac{1}{2}a^\dagger a\rho_c^{'} - \tfrac{1}{2}\alpha(a^\dagger + a)\rho_c^{'} \\ &\quad -\tfrac{1}{2}|\alpha|^2\rho_c^{'} - \tfrac{1}{2}\rho_c^{'}a^\dagger a - \tfrac{1}{2}\alpha\rho_c^{'}(a^\dagger + a) - \tfrac{1}{2}|\alpha|^2\rho_c^{'} \\ &= \mathcal{D}[a]\rho_c^{'} + \tfrac{1}{2}\alpha\left(\rho_c^{'}(a^\dagger - a) - (a^\dagger - a)\rho_c^{'}\right), \end{aligned} \tag{7}$$

and the measurement term is:

$$D\left(-ia\rho_c + i\rho_c a^\dagger - \mathrm{Tr}[-ia\rho_c + i\rho_c a^\dagger]\rho_c\right) D^\dagger \;=\; -i(a+\alpha)\rho_c^{'} + i\rho_c^{'}(a^\dagger+\alpha) - \mathrm{Tr}[-ia\rho_c + i\rho_c a^\dagger]\rho_c^{'} \;=\; -ia\rho_c^{'} + i\rho_c^{'}a^\dagger - \mathrm{Tr}[-ia\rho_c + i\rho_c a^\dagger]\rho_c^{'}. \quad (8)$$

The unitary transform doesn't change the trace, so:

$$\mathrm{Tr}[-ia\rho_c + i\rho_c a^\dagger] = \mathrm{Tr}[-ia\rho_c^{'} + i\rho_c^{'} a^\dagger].$$

Then we have:

$$D(\mathcal{H}[-ia]\rho_c])D^\dagger = \mathcal{H}[-ia]\rho_c^{'}$$

In this "displacement picture", the Stochastic Master equation becomes:

$$d\rho_c^{'} = -\frac{i}{\hbar}\left[H_m - \hbar g(a^\dagger a + \alpha(a+a^\dagger) + |\alpha|^2)x + i\hbar E(a-a^\dagger), \rho_c^{'}\right]dt$$
$$+\gamma\mathcal{D}[a]\rho_c^{'}dt + \frac{1}{2}\gamma\alpha[(a-a^\dagger), \rho_c^{'}]dt + \sqrt{\gamma}\mathcal{H}[-ia]\rho_c^{'}dW.$$

With $\alpha = -2E/\gamma$, two terms in the above equation cancel each other and we end up with the final result:

$$d\rho_c^{'} = -\frac{i}{\hbar}[H_m - \hbar g(a^\dagger a + \alpha(a+a^\dagger) + |\alpha|^2)x, \rho_c^{'}]dt + \gamma\mathcal{D}[a]\rho_c^{'}dt + \sqrt{\gamma}\mathcal{H}[-ia]\rho_c^{'}dW. \quad (9)$$

We can further simplify our analysis by entering the regime:

$$\left|\frac{\langle H_m\rangle}{\gamma}\right| \sim \frac{g(|\alpha|^2+1)|\langle x\rangle|}{\gamma} = \epsilon \ll 1$$

where ϵ will be our small parameter governing the approximation such that we can adiabatically eliminate the off diagonal elements. Basically, this approximation is that the optical damping rate is much larger than the energy in the mechanical mode, such that the photons spend little time in the cavity. This implies that the off diagonal terms have reached their steady state values determined by the mechanical state; ie. the optical modes are slaved to the mechanical modes. We assume that the elements of the cavity mode density matrix in the number basis, $\rho_c^{'nm}$, scale with the small parameter ϵ as $\rho_c^{'nm} \propto \epsilon^{nm}$. Under this assumption, only the three lowest cavity modes remain. The system (cavity + mirror) can be expanded as:

$$\rho_c^{'} = \rho_{00}^{\mathrm{a}}|0\rangle\langle 0| + (\rho_{10}^{\mathrm{a}}|1\rangle\langle 0| + \mathrm{H.c}) + \rho_{11}^{\mathrm{a}}|1\rangle\langle 1| + (\rho_{20}^{\mathrm{a}}|2\rangle\langle 0| + \mathrm{H.c}) + O(\epsilon^3), \quad (10)$$

where ρ_a is the reduced density matrix for the mirror given by:

$$\rho_a = \mathrm{Tr_c}[\rho_{\mathrm{c}}^{'}] = \rho_{00}^{\mathrm{a}} + \rho_{11}^{\mathrm{a}} + O(\epsilon^3). \quad (11)$$

where $\mathrm{Tr_c}$ represents a partial trace over the cavity mode.

The equations of motion for the reduced density matrix elements ρ^a_{ij} are obtained from the master equation for $\rho^{'}_c$. First, we will analyze part of the commutator term in the SME (9) by tracing over the cavity modes.

$$d\rho^{'}_c = -\frac{i}{\hbar}[H_m - hg|\alpha|^2 x, \rho^{'}_c]dt$$

Tracing out the cavity leads to the following, since everything is independent of field operators:

$$d\rho^a = -\frac{i}{\hbar}[H_m - hg|\alpha|^2 x, \rho^a]dt.$$

The coupling term in the commutator of equation (9):

$$d\rho^{'}_c = ig[a^\dagger a x, \rho^{'}_c]dt,$$

will become:

$$d\left(\rho^a_{00}|0\rangle\langle 0| + \rho^a_{10}|1\rangle\langle 0| + \rho^a_{01}|0\rangle\langle 1| + \rho^a_{11}|1\rangle\langle 1| + \rho^a_{20}|2\rangle\langle 0| + \rho^a_{02}|0\rangle\langle 2|\right) = igx(\rho^a_{10}|1\rangle\langle 0| + \rho^a_{11}|1\rangle\langle 1|$$
$$+2\rho^a_{20}|2\rangle\langle 0|)dt - ig(\rho^a_{01}|0\rangle\langle 1| + \rho^a_{11}|1\rangle\langle 1| + 2\rho^a_{02}|0\rangle\langle 2|)xdt$$

after expanding $\rho^{'}_c$ as in equation (10).

In a similar fashion we manipulate the driving term in equation (9):

$$d\rho^{'}_c = ig\alpha[(a + a^\dagger)x, \rho^{'}_c]dt,$$

will become:

$$d\left(\rho^a_{00}|0\rangle\langle 0| + \rho^a_{10}|1\rangle\langle 0| + \rho^a_{01}|0\rangle\langle 1| + \rho^a_{11}|1\rangle\langle 1| + \rho^a_{20}|2\rangle\langle 0| + \rho^a_{02}|0\rangle\langle 2|\right)$$
$$= ig\alpha x(\rho^a_{00}|1\rangle\langle 0| + \rho^a_{10}|0\rangle\langle 0| + \sqrt{2}\rho^a_{10}|2\rangle\langle 0| + \rho^a_{01}|1\rangle\langle 1| + \rho^a_{11}|0\rangle\langle 1| + \sqrt{2}\rho^a_{11}|2\rangle\langle 1| + \sqrt{2}\rho^a_{20}|1\rangle\langle 0| + \rho^a_{02}|1\rangle\langle 2|)dt$$
$$-ig\alpha(\rho^a_{00}|0\rangle\langle 1| + \rho^a_{10}|1\rangle\langle 1| + \rho^a_{01}|0\rangle\langle 0| + \sqrt{2}\rho^a_{01}|0\rangle\langle 2| + \rho^a_{11}|1\rangle\langle 0| + \sqrt{2}\rho^a_{11}|1\rangle\langle 2| + \rho^a_{20}|2\rangle\langle 1| + \sqrt{2}\rho^a_{02}|0\rangle\langle 1|)xdt.$$

So for just the commutator part, we see that after substituting Eq.(10) and then tracing over the cavity modes, all that we are left with are these terms:

$$\begin{aligned}
d\rho^a_{00} &= \mathcal{L}^0_m\rho^a_{00}dt + ig\alpha(x\rho^a_{10} - \rho^a_{01}x)dt, \\
d\rho^a_{10} &= \mathcal{L}^0_m\rho^a_{10}dt + (igx\rho^a_{10} + ig\alpha x\rho^a_{00} + ig\alpha x\sqrt{2}\rho^a_{20} - ig\alpha\rho^a_{11}x)dt, \\
d\rho^a_{11} &= \mathcal{L}^1_m\rho^a_{11}dt + (ig\alpha x\rho^a_{01} - ig\alpha\rho^a_{10}x)dt, \\
d\rho^a_{20} &= \mathcal{L}^0_m\rho^a_{20}dt + (2igx\rho^a_{20} + ig\alpha\sqrt{2}x\rho^a_{10})dt,
\end{aligned} \tag{12}$$

where we define:

$$\mathcal{L}^l_m\rho^a_{ij}dt \equiv -\frac{i}{\hbar}[H_m - \hbar g(|\alpha|^2 + l)x, \rho^a_{ij}]dt.$$

Now, looking at the damping terms of Eq.(9) we can see that we can write it like this:

$$\begin{array}{rcl} d\rho^a_{00} &=& \gamma\rho^a_{11}dt \\ d\rho^a_{10} &=& -\frac{1}{2}\gamma\rho^a_{10}dt \\ d\rho^a_{11} &=& -\gamma\rho^a_{11}dt \\ d\rho^a_{20} &=& -\gamma\rho^a_{20}dt \end{array} \tag{13}$$

in a very similar fashion.

The dW terms are derived in a completely analogous fashion, except that one should note that:

$$\mathrm{Tr}[-\mathrm{i}(\mathrm{a}-\mathrm{a}^\dagger)\rho_\mathrm{c}^{'}] = \mathrm{Tr}[-\mathrm{i}\rho^\mathrm{a}_{10} + \mathrm{i}\rho^\mathrm{a}_{01}] + O(\epsilon^3). \tag{14}$$

Finally, if we combine equations (11), (12), (13) and the dW terms we get the equations of motion for ρ^a_{ij} as follows:

$$d\rho^a_{00} = \mathcal{L}^0_m\rho^a_{00}dt + ig\alpha(x\rho^a_{10} - \rho^{a\dagger}_{10}x)dt + \gamma\rho^a_{11}dt - i\sqrt{\gamma}(\rho^a_{10} - \rho^{a\dagger}_{10} - \mathrm{Tr}[\rho^\mathrm{a}_{10} - \rho^{\mathrm{a}\dagger}_{10}]\rho^\mathrm{a}_{00})\mathrm{dW} + O(\epsilon^3), \tag{15}$$

$$\begin{array}{rcl} d\rho^a_{10} &=& \mathcal{L}^0_m\rho^a_{10}dt - \frac{\gamma}{2}\rho^a_{10}dt + ig[x(\alpha\rho^a_{00} + \rho^a_{10} + \sqrt{2}\alpha\rho^a_{20}) \\ && -\alpha\rho^a_{11}x]dt - i\sqrt{\gamma}(\sqrt{2}\rho^a_{20} - \rho^a_{11} - \mathrm{Tr}[\rho^\mathrm{a}_{10} - \rho^{\mathrm{a}\dagger}_{10}]\rho^\mathrm{a}_{10})\mathrm{dW} + O(\epsilon^3), \end{array} \tag{16}$$

$$d\rho^a_{11} = \mathcal{L}^1_m\rho^a_{11}dt + ig\alpha(x\rho^{a\dagger}_{10} - \rho^a_{10}x)dt - \gamma\rho^a_{11}dt + i\sqrt{\gamma}\mathrm{Tr}[\rho^\mathrm{a}_{10} - \rho^{\mathrm{a}\dagger}_{10}]\rho^\mathrm{a}_{11}\mathrm{dW} + O(\epsilon^3), \tag{17}$$

$$d\rho^a_{20} = \mathcal{L}^0_m\rho^a_{20}dt - \gamma\rho^a_{20}dt + igx(2\rho^a_{20} + \sqrt{2}\alpha\rho^a_{10})dt + i\sqrt{\gamma}\mathrm{Tr}[\rho^\mathrm{a}_{10} - \rho^{\mathrm{a}\dagger}_{10}]\rho^\mathrm{a}_{20}\mathrm{dW} + O(\epsilon^3). \tag{18}$$

To get a stochastic master equation for the cavity reduced density matrix $\rho_a = \rho^a_{00} + \rho^a_{11}$, we will adiabatically eliminate the off-diagonal elements ρ^a_{10} and ρ^a_{20}. To do this, we will take advantage of the different times scales, between the mechanical and optical modes, and assume that the off diagonal terms will damp to their steady state values quickly, such that we ignore their fluctuation and just set them to their steady-state values. We only keep the damping term and drop the other two terms proportional to ρ^a_{20}. In doing so we get:

$$d\rho^a_{20} = -\gamma\rho^a_{20}dt + i\sqrt{2}g\alpha x\rho^a_{10}dt + i\sqrt{\gamma}\mathrm{Tr}[\rho^\mathrm{a}_{10} - \rho^{\mathrm{a}\dagger}_{10}]\rho^\mathrm{a}_{20}\mathrm{dW}.$$

Therefore, the steady-state solution is:

$$\rho^a_{20} = i\sqrt{2}\left(\frac{g\alpha}{\gamma}\right)x\rho^a_{10} + O(\epsilon^3). \tag{19}$$

In a similar fashion we can eliminate the terms that are lowest order in ϵ, ρ^a_{10} and ρ^a_{20} on therir own, to get:

$$d\rho^a_{10} = -\frac{\gamma}{2}\rho^a_{10}dt + ig[x(\alpha\rho^a_{00}) - \alpha\rho^a_{11}x]dt - i\sqrt{\gamma}(\sqrt{2}\rho^a_{20} - \rho^a_{11} - \mathrm{Tr}[\rho^\mathrm{a}_{10} - \rho^{\mathrm{a}\dagger}_{10}]\rho^\mathrm{a}_{10})\mathrm{dW}$$

and further more we will then drop the stochastic part (which is also not leading order in ϵ) and assume that this term has reached steady state to get:

$$\rho^a_{10} = 2i\left(\frac{g\alpha}{\gamma}\right)[x\rho^a_{00} - \rho^a_{11}x] + O(\epsilon^3). \tag{20}$$

Substitute eqns. (19) and (20), into eqns. (15) and (17), and then add these two equations above to get an approximate equation of motion for $d\rho_a = d\rho^a_{00} + d\rho^a_{11}$ of the form:

$$d\rho_a = -\frac{i}{\hbar}[H_m - \hbar g|\alpha|^2 x, \rho_a]dt + 2k\mathcal{D}[x]\rho_a dt + \sqrt{2k}\mathcal{H}[x]\rho_a dW. \tag{21}$$

where the measurement constant $k = 2g^2|\alpha|^2/\gamma$. Notice the arguments of the superoperators above; their arguments are now the position of the mirror and not the annihilation operator of the cavity mode.

An important point in deriving the above equations is that the term proportional to $\rho^a_{11}x$ in eqn.(20) is of third order in ϵ, so changing the sign of that term doesn't effect our calculation to our desired order of this approximation. Once we get to this form of the Stochastic Master equation, we can get the corresponding Stochastic Schrödinger equation by expand $d\rho = (|\psi\rangle + d|\psi\rangle)(\langle\psi| + d\langle\psi|) - |\psi\rangle\langle\psi|$ as usual:

$$d|\psi\rangle = -\frac{i}{\hbar}(H_m - \hbar g|\alpha|^2 x)|\psi\rangle dt - k(x - \langle x\rangle)^2|\psi\rangle dt + \sqrt{2k}(x - \langle x\rangle)|\psi\rangle dW. \tag{22}$$

This is the main result. What we have derived up to this point is the Stochastic Schrödinger equation which is what our program will solve. To conclude we will give a synopsis of the derivation here: after transforming our Stochastic Master equation into the displacement picture, so that the steady state of the cavity mode will be close to the vacuum, then we made the adiabatic approximation. This allows one to let the off diagonal terms damp to their steady state values, due to the large decay rate of our cavity, to then be substituted back into the diagonal terms equations of motion to which leaves us with just a Stochastic Master equation for just the mirror motion. We can then transform that equation into a Stochastic Schrödinger equation to then be analyzed by our program which will be describe in the next few sections.

3 The measurement record

The measurement signal, or photocurrent, is given by [3]:

$$dQ = \beta[-\gamma\langle i(a - a^\dagger)\rangle)dt + \sqrt{\gamma}dW. \tag{23}$$

where β is a coefficient determining the strength of the measurement. Furthermore we can simplify the first term by noting:

$$\langle -i(a - a^\dagger)\rangle = \mathrm{Tr}[-\mathrm{i}(\rho^{\mathrm{a}}_{10} - \rho^{\mathrm{a}\dagger}_{10})] = 4\mathrm{g}\alpha\langle \mathrm{x}\rangle/\gamma. \tag{24}$$

Substitute eqn. (24) into eqn. (23), we will then have:

$$dQ = \beta[2\sqrt{2\gamma k}\langle x\rangle dt + \sqrt{\gamma}dW.$$

We will further scale this photocurrent and use:

$$dV = \langle x \rangle dt + \frac{dW}{\sqrt{8k}} \tag{25}$$

as our measurement record which will then be used in our feedback system in the program.

4 Motion of mean values

Now, our Hamiltonian is:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \hbar g|\alpha|^2 x,$$

and if we follow a similar approach as that done in the class lecture notes, for Physics 686 at the University of Oregon which is written by Professor Daniel Steck, on page 142, where the evolution of the means for the damped quantum harmonic oscillator were calculated, we get an evolution for our mean mirror position and mean mirror momentum of:

$$d_t\langle x \rangle = -\frac{i}{\hbar}\langle [x, H] \rangle = \frac{\langle p \rangle}{m},$$

and

$$d_t\langle p \rangle = -\frac{i}{\hbar}\langle [p, H] \rangle = -m\omega^2\langle x \rangle + \hbar g|\alpha|^2.$$

Decoupling these two equations gives us:

$$d_t^2\langle x \rangle = -\omega^2\langle x \rangle + \frac{\hbar g}{m}|\alpha|^2,$$

and

$$d_t^2\langle p \rangle = -\omega^2\langle p \rangle.$$

where we see that the equation of motion for the mean position is that of a shifted harmonic oscillator and the evolution of the momentum is just that of a typical harmonic oscillator.

5 Rescaling

For programming convenience, we scaled the equations so that variables for our program are dimensionless. If one wants the real values for these scaled values one would just use the definitions here and the values chosen for the scaling to convert back. Variables that are used are as follows:

$$\tilde{x} = \frac{x}{\sqrt{\frac{\hbar}{m\omega_m}}}, \qquad \tilde{p} = \frac{p}{\sqrt{m\hbar\omega_m}}, \qquad \tilde{t} = t\omega_m$$

The variables without tilde are the real unscaled values. We can define new scaling constants as:

$$x_0 = \sqrt{\frac{\hbar}{m\omega_m}}, \qquad p_0 = \sqrt{m\hbar\omega_m}$$

Other scaled parameters used in the program are:

$$\tilde{g} = gx_0/\omega_m, \qquad \tilde{\gamma} = \gamma/\omega_m, \qquad \tilde{k} = 2\tilde{g}^2|\alpha|^2/\tilde{\gamma}, \qquad d\tilde{W} = \sqrt{\omega_m}dW.$$

Under this enlightenment, the stochastic Schrödinger equation becomes:

$$d|\psi\rangle = -i\left(\frac{\tilde{p}^2}{2} + \frac{\tilde{x}^2}{2} - \tilde{g}|\alpha|^2\tilde{x}\right)d\tilde{t}|\psi\rangle - \tilde{k}(\tilde{x} - \langle\tilde{x}\rangle)^2|\psi\rangle d\tilde{t} + \sqrt{2\tilde{k}}(\tilde{x} - \langle\tilde{x}\rangle)|\psi\rangle d\tilde{W}. \tag{26}$$

The measurement record will become:

$$d\tilde{V} = \langle\tilde{x}\rangle d\tilde{t} + \frac{d\tilde{W}}{\sqrt{8\tilde{k}}}.$$

The equation of motion for the mean value of x becomes:

$$d_{\tilde{t}}^2\langle\tilde{x}\rangle = -\langle\tilde{x}\rangle + \tilde{g}|\alpha|^2.$$

Here, if we add the feedback: $|\alpha|^2 = (\alpha_0 - 2\dot{\tilde{x}})/\tilde{g}$

it will case the oscillator to be critically damped. This is what we are aiming for. We then have:

$$d_{\tilde{t}}^2\langle\tilde{x}\rangle = -\langle\tilde{x}\rangle + \alpha_0 - 2\dot{\tilde{x}}$$

for the evolution of the mean. In our program, we use the parameters from the Wang Group of Young-Shin Parks's microspheres at University of Oregon. They are:

Effective mass:

$$m_{eff} = 1.244 \times 10^{-16}\text{kg}$$

which has been determined to be between $\frac{1}{2}$ to $\frac{1}{3}$ of the total mass of the microsphere.

Mechanical frequency of the sphere's lowest order vibrations:

$$\omega_m = 2\pi \times 100\text{MHz}$$

Optical wavelength:

$$\lambda_O = 632\text{nm}$$

Cavity length:

$$L = 90\mu m$$

Coupling constant:

$$g = \frac{\omega_O}{L} = 10^{19}\text{s}^{-1}\text{m}^{-1}$$

Cavity Mode linewidth:

$$\Delta\omega = 2\pi \times 50\text{MHz}$$

Decay rate:

$$\gamma = \Delta\omega = 3 \times 10^8\text{Hz}$$

Optical power:

$$P = mW$$

Using the values above, we calculate the parameters for our program:

$$\tilde{\gamma} \approx 1$$

$$\tilde{g} \approx 10^{-3}$$

$$|\alpha|^2 = 10^5$$

6 Programming and Feedback

The program code used to solve the reduced cavity-mirror system is based on the split operator evolver, but also makes use of the stochastic differential equation solver modules. It is very similar to the code used to solve the quantum feedback problem on assignment 2. Each time step is broken into three sub steps, the first using the SDE modules, which neglect the momentum part of the evolution, for half of the time step. The second part consists of a fourier transform of the wave function to momentum space, an application of the exponential propagation operator for the momentum, and a back transform to position space. After a second half time step with the SDE modules the wave function represents an approximate solution of the system specified by the potential which was entered into the module sderk_support.f90.

To implement the feedback the measurement record is also modeled in the main program code. After each wave function evolution step the stochastic jump is saved and used to calculate the measurement record as described above. This signal is then sent into a low pass filter modeled by using the low pass differential equation and calculating successive values using the preceding value and the current input value.

$$V = (1 - w3dB * dt) * Vold + w3dB * measurement$$

The parameter $w3dB$ corresponds to the filter's cutoff frequency whose value is specified in a parameter file. By setting it lower than the oscillator frequency most of the high frequency noise should be filtered out without losing too much information about the measured quantity. By using a much lower cutoff a maximal phase lag can be achieved. Taking the filtered signal's derivative in this case reproduces the actual average position pretty well as seen in Fig. 4. It might still be off by some factor which can be determined empirically. We spent sometime here trying to make this as accurate as possible so we had a good number to feedback. We tried a few different methods to get our final result. The main thing that seemed to make the program work well was the size of the array that would calculate the derivative. With this knowledge of this reconstructed quantity, $d_t\langle x\rangle$, the feedback can be implemented.

The feedback is intended to cool the mechanical motion of the mirror. This could be achieved by altering parameters in the system Hamiltonian during evolution. In our case this parameter is the laser power, which is related to the coherent state number α. This parameter can be changed in such a way that it represents a damping term in the equation of motion of the mean position as seen earlier. With perfect knowledge of the mean position critical damping can be achieved which is the case that is aimed for. This case was observed by "cheating" and using the actual value of the mean position and not the filtered measurement. After having seen that the cooling algorithm could be successful the task was to achieve an accurate enough measurement record to reconstruct $d_t\langle x\rangle$. This was achieved by setting the filter's cutoff frequency very low as mentioned above and increasing the laser power offset, the latter causing the program to become increasingly unstable. This problem could be fixed by decreasing the time step size or increasing the number of sub steps for the sderk solver. There was a trade off between accuracy of measurement and program run time.

Figures 2 and 3 show the decreasing energy after feedback kicks in at one scaled time unit. The feedback algorithm obviously cools the mirror successfully. Averaging the total energy over a time period starting at 1.5 and ending at 6 scaled units led to an average oscillator number of about 5.43.

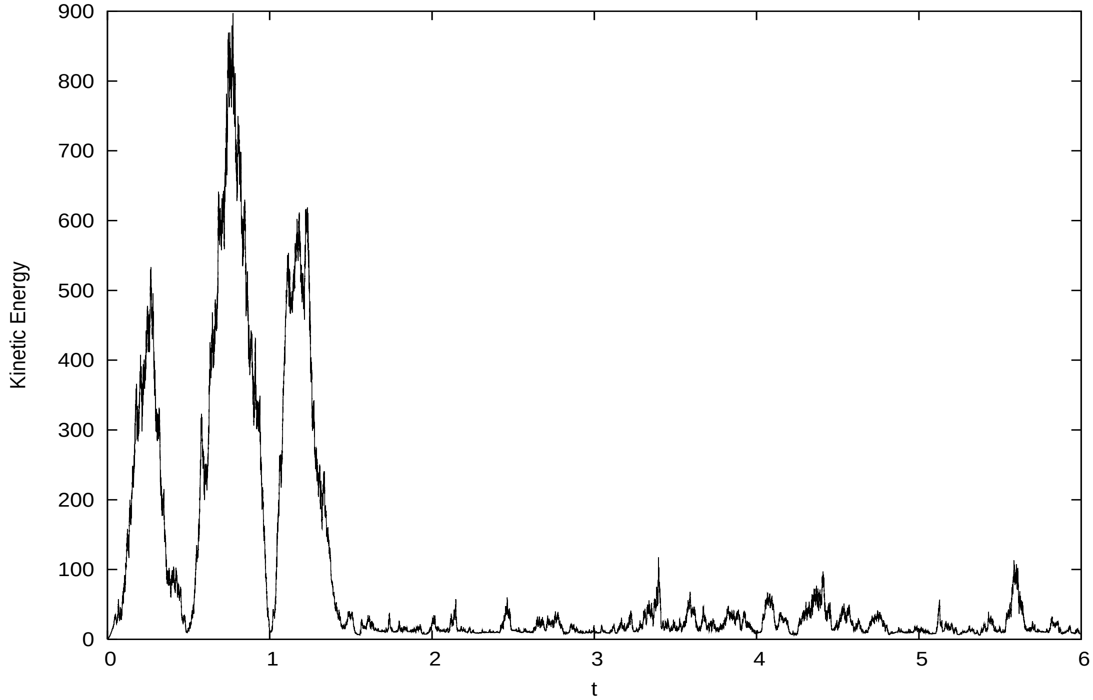

Figure 2: The Kinetic Energy as a function of time. Feedback starts after one scaled time unit. After being critically damped the noise from the measurement signal dominates and we lose the mirror only to get it back later.

7 Conclusion

Our group studied and simulated the feedback control of a mechanical oscillator using continuous measurement. However, we had to first get a master equation just for the mirror to simplify our system. Once this was done we were able to implement a feedback algorithm, to cool the mechanical motion of a movable end mirror that forms a cavity, via radiation pressure. Simulation of the measurement record modeling the homodyne photocurrent allowed to gain information about the mirror's mean position using a low pass filter. This made it possible to apply our quantum feedback control algorithm to cool the mechanical motion.

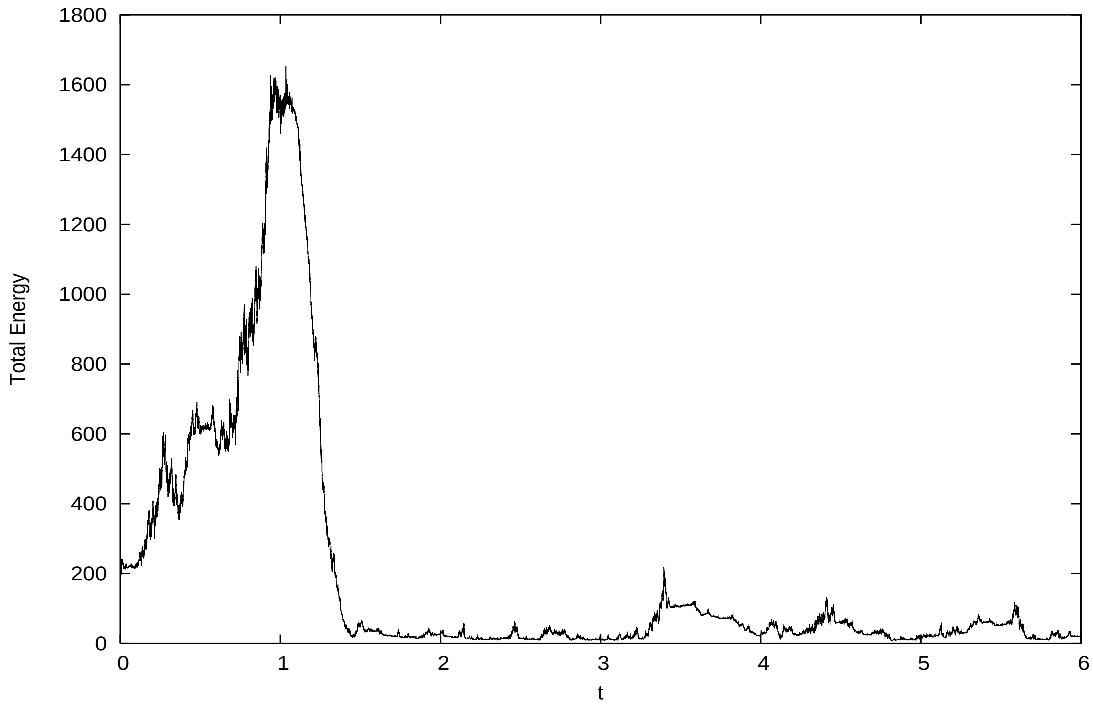

Figure 3: The Total Energy as a function of time. Feedback starts after one scaled time unit. After being critically damped the noise from the measurement signal dominates and we lose the mirror only to get it back later just as before.

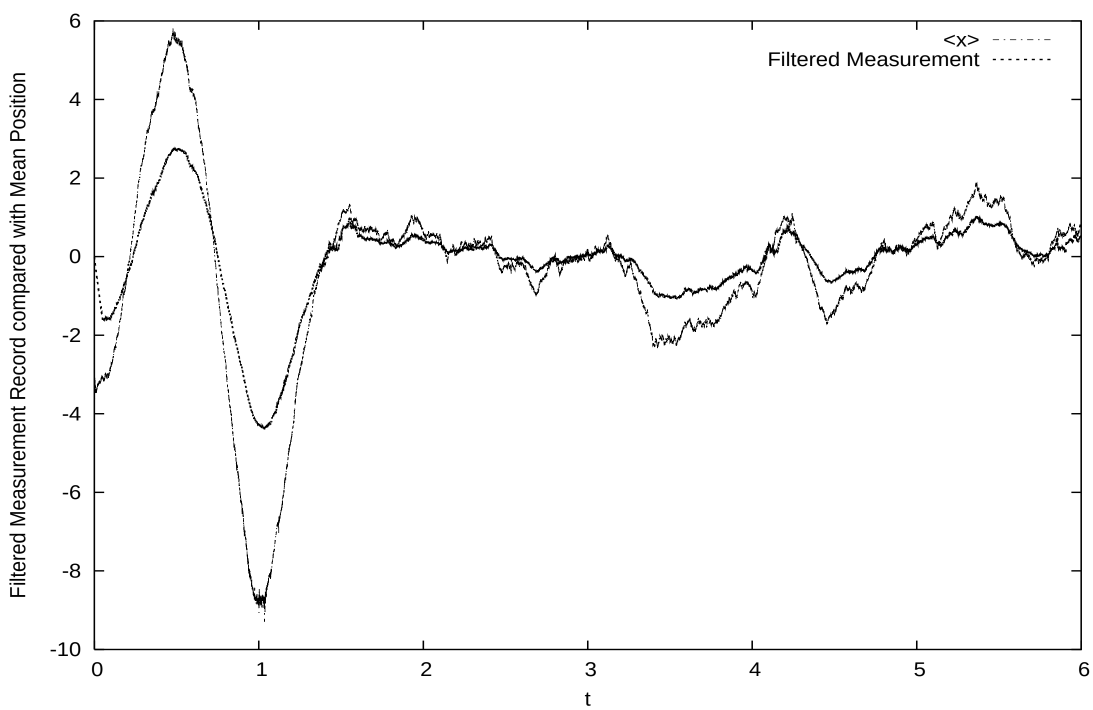

Figure 4: This is a plot of the mean position compared with our calculated value for the mean position from the measurement record after it has been filtered through a low pass filter.

References

[1] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T.J. Kippenber,"Theor of ground state cooling of a mechanical oscillator using dynamical back-action", arXiv:cond-mat/0702113v1 [cond-mat.mes-hall]

[2] A.C. Doherty, K. Jacobs,"Feedback control of quantum systems using continuous state estimation", Phys. Rev. A, Vol. 60, No. 4, 1999

[3] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **47**, 642 (1993)