2

FLOW OF GASES THROUGH TUBES AND ORIFICES

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The nature of gas flow in pipes and ducts changes with the gas pressure and its description is generally divided into three parts or *regimes*. The flow dynamics are characterized by λ, the molecular mean free path, in relation to some characteristic dimension such as the diameter of a pipe. The flow regime cannot be determined from the mean free path alone but only from the relation of this parameter to the characteristic dimension. The relation is known as the *Knudsen number*, defined as*

$$\mathrm{Kn} = \frac{\lambda}{d}. \tag{2.1}$$

Three regimes are generally identified:

1. *Free Molecular Flow.* The mean free path is of the same order as, or greater than, the characteristic dimension (the range of relatively large Knudsen numbers), and gas dynamics are dominated by molecular collisions with the walls of the retaining vessel or pipe.

2. *Continuum Flow.* The mean free path is small compared with the characteristic dimension (the range of small Knudsen numbers), and intermolecular collisions are much more frequent than wall collisions. In this regime the properties of the gas

* In the literature the Knudsen number may be variously defined, for a cylindrical tube, as λ/d, d/λ, λ/R, or R/λ.

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(temperature, density, flow velocity) do not vary significantly over several mean free paths and the gas can be considered a continuous medium. The gas dynamics are therefore described and analysed hydrodynamically. Flow in this regime is often referred to as *viscous flow*, although there are circumstances (such as flow through short ducts) in which viscosity plays no part.

3. *Transitional Flow.* The transition between continuum and free molecular flow occurs at intermediate values of the Knudsen number where both wall collisions and intermolecular collisions are influential in determining the flow characteristics.

Expressed in terms of pressure and characteristic dimension the Knudsen number is

$$\mathrm{Kn} = \frac{\eta}{Pd}\sqrt{\frac{\pi}{2}\frac{R_0 T}{M_m}}. \tag{2.2}$$

For air at 20°C, with d in mm and P in mbar we obtain

$$\mathrm{Kn} = \frac{0.066}{Pd}. \tag{2.3}$$

Table 2.1 shows the generally accepted range of Knudsen numbers for the three regimes. There is no sharp transition between the regimes, and somewhat different values may be quoted by different authors. The gas factor F_g in the table, used to correct for different gases, is the ratio of the mean free path for air to that of the gas under consideration (at the same pressure and temperature) and can be calculated from

$$F_g = \frac{\lambda_{\mathrm{air}}}{\lambda_{\mathrm{gas}}} = \frac{\eta_{\mathrm{air}}}{\eta_{\mathrm{gas}}}\sqrt{\frac{M_{\mathrm{gas}}}{M_{\mathrm{air}}}}. \tag{2.4}$$

Values of the gas factor for a number of common gases are shown in Table 2.2.

Applications of vacuum technology range from the lowest pressures attainable ($< 10^{-14}$ mbar) through to around atmospheric pressure, so that all of the regimes described are of interest to workers in this field. One of the main aims of this chapter is to enable calculation of flow under as wide a range of conditions as possible, so that a large number of flow equations is presented. The widespread availability of scientific calculators, personal computers, and mathematical software means that some of the more cumbersome formulae are considerably less daunting than in times past. However, the "back of an envelope" is still a much favored tool of scientists and engineers and rough calculations are often sufficient, so that approximations will be given wherever possible. Equations for the molecular, continuum and transitional

Table 2.1. Flow Regimes versus Knudsen Number and Pressure

Regime	Kn (λ/d)	P (mbar), d (mm)
Molecular	$\mathrm{Kn} > 0.5$	$PdF_g < 0.133$
Transitional	$0.5 > \mathrm{Kn} > 0.01$	$0.133 < PdF_g < 6.6$
Continuum	$\mathrm{Kn} < 0.01$	$PdF_g > 6.6$

Table 2.2. Properties of Some Common Gases at 20°C

Gas	Relative Molecular Mass	Viscosity $Pa \cdot s \times 10^{-6}$	Viscosity Ratio (Air/Gas)	F_g
H_2	2	8.8	2.07	0.543
He	4	19.6	0.929	0.345
H_2O (vapor)	18	9.7	1.88	1.48
N_2	28	17.6	1.03	1.02
Air	29	18.2	1	1
O_2	32	20.4	0.892	0.937
Ar	40	22.3	0.818	0.959
CO_2	44	14.7	1.24	1.53

regimes are discussed in Sections 2.2, 2.3, and 2.4, respectively. For the most part, derivations are not given since there are textbooks and papers where the basic theories are discussed extensively; several references are listed in each section for the reader interested in studying the subject in more detail.

All equations are written at least once in the text in SI units; where numerical coefficients are given, the units used are stated.

Except for the discussion of adiabatic compressible flow, it is generally assumed throughout this chapter that isothermal conditions apply.

2.1. FLOW CONDUCTANCE, IMPEDANCE, AND GAS THROUGHPUT

In the field of vacuum science and technology it is common practice to express gas flow rate as *throughput* in pressure–volume units. The symbol $\dot{Q}$ is normally used and the throughput of gas at a particular pressure is then

$$\dot{Q} = \frac{d(PV)}{dt} = P\frac{dV}{dt}. \tag{2.5}$$

If the volumetric flow rate is due to a pump

$$\dot{Q} = P\frac{dV}{dt} = PS, \tag{2.6}$$

where S is the speed (or volumetric rate) of the pump at the pressure P.

The pumping speed available at a chamber will be affected by restriction due to connecting pipework. One of the most common problems in vacuum technology is to estimate the loss in speed due to such restrictions (system design is covered in Chapter 9).

Knudsen [1] first introduced the notion of a pipe as an impedance or resistance in the electrical sense and Dushman [2] introduced the concept of *conductance*, which is defined by the relation

$$C = \frac{\dot{Q}}{P_u - P_d}, \tag{2.7}$$

where P_u is the upstream pressure and P_d is the downstream pressure. These pressures normally refer to values in (perhaps notional) plenums at the entrance and exit of a duct or a system fitting such as a valve. Gas flow conductance is thus analogous to electrical conductance, with pressure difference being the analogue of voltage difference and $\dot{Q}$ the analogue of current. The reciprocal of conductance (resistance or impedance, $Z = 1/C$) could equally well be used; however, conductance has come into common usage in vacuum technology mainly because of its intuitive relation to volume flow rate and pumping speed.

Applying this concept to a set of pipes or components in series, the net conductance is found from

$$\frac{1}{C_n} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots. \tag{2.8}$$

The net speed of a pump in series with a component or pipe is found in a similar way:

$$\frac{1}{S_n} = \frac{1}{S} + \frac{1}{C}. \tag{2.9}$$

In practice it is seldom quite so easy as these equations imply. Some care is needed with combinations of components, and this is discussed in Section 2.2.10 in relation to molecular flow. In continuum flow, conductance depends in a complicated way on the flow conditions, and a better approach is to calculate the pressure ratio (K_p) across a component or series of components.

It is usually assumed that continuity applies through a system; that is, the throughput is the same through all sections. This will be the case as long as sufficient time has elapsed (from opening a valve or starting a pump for example) and there are no temperature differences between the points of interest. In many common situations, steady conditions are a reasonable assumption. (Some cases of unsteady molecular flow are covered in Section 2.2.11.) If P_d is the inlet pressure to a pump of speed S which is connected via a pipeline or component to a chamber, then, assuming steady conditions, the speed (S_n) and pressure (P_u) at the chamber are simply related by

$$P_u S_n = P_d S, \tag{2.10}$$

so that

$$S_n = \frac{P_d}{P_u} S = \frac{S}{K_p}. \tag{2.11}$$

In this way, the net pumping speed can be found if the pressure ratio can be calculated.

From the definition of conductance

$$\dot{Q} = (P_u - P_d)\, C. \tag{2.12}$$

Dividing through by the downstream pressure P_d and rearranging gives

$$K_p = 1 + \frac{S}{C}. \tag{2.13}$$

This calculation is often easier than taking the reciprocal of a sum of reciprocals, and K_p is the factor by which the pumping speed is reduced.

2.2. MOLECULAR FLOW

One of the fundamental assumptions in the derivations of molecular flow conductance is that molecules scatter from a surface according to a cosine distribution. This is also referred to as diffuse or random scattering and means that there are no favored directions. A scattered molecule has the same probability of emerging in any direction, and this is unrelated to its direction of incidence. There are special circumstances in which nondiffuse scattering may occur, but for microscopically rough surfaces the diffuse scattering law is well established theoretically and experimentally [3–5].

Figure 2.1 shows the distribution of molecules emerging from an aperture and from tubes of various lengths. The lengths of the vectors are proportional to the number of molecules emerging in that direction. It is noticeable that the longer the tube, the more heavily weighted is the emerging flux to the tube axis. This is an indication of what occurs inside a tube. An observer close to the entrance (looking upstream) will see the entrance plane as a diffuse source. Deep inside the tube an observer will see a perturbed flux which is peaked toward the axis. This is often referred to as the beaming effect of a tube.

The aperture in Fig. 2.1 acts as a plane cosine emitter, and the molecular flux shows a spherical distribution. At small angles to the plane the flux of molecules is reduced,

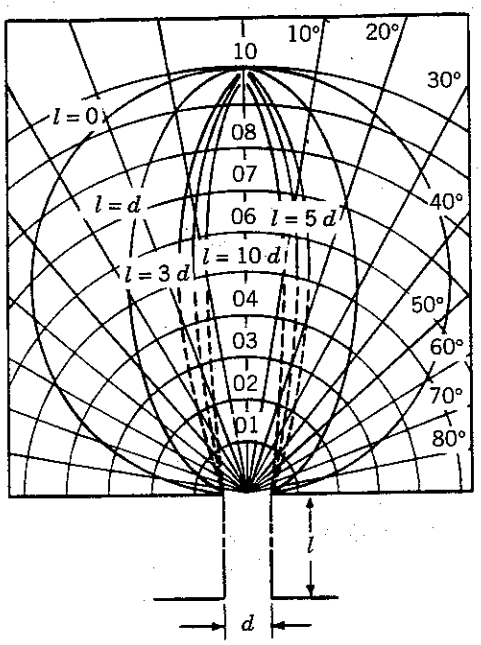

Fig. 2.1. Angular distribution of molecules exiting tubes of various length-to-diameter ratios. Reproduced with permission from L. Valyi, *Atom and Ion Sources*, p. 86. Copyright 1977, Akadémial Kiadó, Budapest.

compared with larger angles. This is not because there is a preferred direction but is a consequence of the angle of view: The emitting plane does not appear "dimmer," simply smaller.

In the molecular regime, solution of gas flow problems can be reduced to finding the conductance of the elements involved since conductance is independent of pressure or flow conditions. The derivation of conductance, by theoretical or analyticostatistical methods, assumes that molecules arrive at the entrance plane of a duct from a chaotic gas, so that the entrance plane effectively behaves as a diffuse (cosine) emitter. When vacuum components are connected in series, this may not be the case and some correction is needed; this will be covered in Section 2.2.10.

Clausing [6] first introduced the concept of transmission probability, denoted by α. If N_2 molecules arrive at the entrance plane of a duct then the number of these which reach the exit plane is $N_2\alpha$, and $N_2(1-\alpha)$ return to the entrance. Similarly, of N_1 molecules striking the exit plane (from a downstream chamber), $N_1\alpha$ reach the entrance. The *net* flux of molecules from entrance to exit is then $(N_2 - N_1)\alpha$. Although proportional to the pressure difference across the duct, the net flux is not driven by a pressure difference; it actually consists of two independent fluxes, and there is not a flow in the usual sense of the word. Some of the molecules which enter the duct will return to the entry plane after one or more wall collisions. The flow dynamics are thus very different to the continuum flow case in which all molecules crossing the entrance plane will leave the exit (apart from the possibility of back diffusion which can occur in some circumstances).

Expressions for conductance are usefully formulated in terms of transmission probability, so that the conductance of a duct (or other component) is given by the entrance aperture conductance multiplied by the transmission probability

$$C_m = C_a\alpha. \tag{2.14}$$

2.2.1. Conductance of an Aperture

The molecular flow conductance of a thin aperture is directly related to the rate of impingement of molecules over the aperture area A (discussed in Chapter 1):

$$C_a = A\sqrt{\frac{R_0T}{2\pi M_m}}, \tag{2.15}$$

where R_0 is the universal gas constant, T is the thermodynamic (or absolute) temperature, and M_m is the molar mass (e.g., 0.028 kg/mole for nitrogen).

This gives the familiar result that the molecular flow conductance of an aperture for air at 20°C is 11.6 liters per second per square centimeter.

For air at 20°C, Eq. (2.15) can be written in the convenient form

$$C_a = k_aA, \tag{2.16}$$

and Table 2.3 gives values of the constant k_a for several combinations of commonly used units.

Table 2.3. Aperture Conductance for Air at 20°C in Various Units

	$C_a = k_a A$	
C_a	k_a	A
$m^3 \cdot s^{-1}$	115.6	m^2
liter $\cdot s^{-1}$	11.56	cm^2
liter $\cdot s^{-1}$	0.1156	mm^2
$m^3 \cdot h^{-1}$	0.4163	mm^2
cfm	0.245	mm^2
cfm	158.1	$in.^2$
liter $\cdot s^{-1}$	74.62	$in.^2$

2.2.2. General Considerations for Long Ducts

Molecular flow in long ducts was first studied experimentally and theoretically by Knudsen [7]. He deduced a general relationship for a long duct of length l, varying cross-sectional area A and perimeter B, which can be written as

$$C_m = \frac{4}{3} v_a \left[\int \frac{B}{A^2} \, dl \right]^{-1}, \tag{2.17}$$

where v_a is the mean thermal velocity of molecules.

However, as discussed by Steckelmacher [8, 9], Eq. (2.17) gives the correct result only in the case of a long cylindrical tube and leads to erroneous results for all other cross sections.

A correct expression was derived by Smoluchowski [10] which may be written

$$C_{ml} = \frac{v_a}{8l} \int_s \int_{-\pi/2}^{+\pi/2} \frac{1}{2} \rho^2 \cos\theta \, d\theta \, ds, \tag{2.18}$$

where ρ is a chord making an angle θ with the normal to the perimeter s. Expressions for a number of different cross sections have been derived from Eq. (2.18).

2.2.3. General Considerations for Short Ducts

For small values of l it is clear that the long duct relation will give values for conductance which are too high. As the length tends to zero the conductance apparently tends to infinity. Reasoning from the point of view that the entrance of a duct can be considered as a vacuum circuit element with resistance $Z_a = 1/C_a$ in series with the duct proper (regarded as a "long" duct), of resistance $Z_{ml} = 1/C_{ml}$, the net conductance is [applying Eq. (2.8)]

$$\frac{1}{C_m} = \frac{1}{C_{ml}} + \frac{1}{C_a}. \tag{2.19}$$

Using this principle, the net transmission probability of a short duct then becomes

$$\alpha = \frac{\alpha_l}{1 + \alpha_l}, \tag{2.20}$$

where α_l is the long duct transmission probability.

This principle was originally applied by Dushman [2] to short circular cross-sectional tubes as an *approximate* method of correcting for the end effect. A similar logic is often applied to short ducts of other cross sections. The maximum error for a cylindrical tube is about 12% (too high). Errors of this order are expected for blocky cross sections, but for other cross sections the errors may be more serious. In the case of narrow rectangular ducts the errors can be greater than 50%. The most accurate results are obtained using transmission probabilities which have been derived for a number of different shapes either theoretically or via Monte-Carlo methods.

Transmission probability data for cylindrical tubes, from the results of Cole [11], are shown in Table 2.5. It is apparent that the transmission probability *for a unit length* increases as the length of the duct increases. Consider a unit length $l/d = 1$ for which $\alpha = 0.514$. For two unit lengths ($l/d = 2$) the transmission probability, expected from the shorter length, would be 0.257, whereas the actual value is 0.357. At 10 unit lengths the transmission probability is almost twice the expected value. This reflects the effect of the random molecular distribution near the entrance compared with the beamed distribution which evolves further down the tube.

2.2.4. Tube of Uniform Circular Cross Section

Long Tubes. The familiar expression for the conductance of a long cylindrical tube of diameter d was first presented by Knudsen [7] in 1909:

$$C_{ml} = A\sqrt{\frac{R_0 T}{2\pi M_m}}\cdot\frac{4d}{3l} = \frac{d^3}{3l}\sqrt{\frac{\pi R_0 T}{2M_m}}. \tag{2.21}$$

It can be seen from Eq. (2.21) that the transmission probability for a long cylindrical tube is

$$\alpha_c = \frac{4d}{3l}. \tag{2.22}$$

Berman [12] derived the solution as an asymptotic expansion, the first four terms of which are

$$\alpha_c = \frac{4d}{3l} - \frac{1}{2}\left(\frac{d}{l}\right)^2 \ln\left(\frac{2l}{d}\right) - \frac{91}{72}\left(\frac{d}{l}\right)^2 + \frac{4}{3}\left(\frac{d}{l}\right)^3 \ln\left(\frac{2l}{d}\right)\cdots. \tag{2.23}$$

For $l \gg d$ this reduces to Eq. (2.22) (which needs significant correction for $l < 50\,d$).

Table 2.4 lists values of long tube and aperture conductance for a number of gases.

Table 2.4. Conductance of Long Cylindrical Tubes and Apertures for Air at 20°C

Gas	Relative Molecular Mass	C_{ml} (l/d^3) liter·s^{-1} (mm)	C_a/A liter·s^{-1} (mm)	C_a/d^2 liter·s^{-1} (mm)
H_2	2	0.461	0.440	0.346
He	4	0.326	0.311	0.245
Air	29	0.121	0.116	0.0908
Ar	40	0.103	0.0985	0.0773

Short Tubes. Transmission probabilities for short tubes derived by Cole [11] are shown in Table 2.5. Berman [12] presented equations for the direct calculation of α for any length.

Table 2.5. Transmission Probabilities for Cylindrical Tubes

l/d	α (Cole [11])	α [Eq. (2.20)]	% Error
0.05	0.952399	0.963855	− 1.20
0.15	0.869928	0.898876	− 3.33
0.25	0.801271	0.842105	− 5.10
0.35	0.743410	0.792079	− 6.55
0.45	0.694044	0.747664	− 7.73
0.5	0.671984	0.727273	− 8.23
0.6	0.632228	0.689655	− 9.08
0.7	0.597364	0.655738	− 9.77
0.8	0.566507	0.625000	− 10.33
0.9	0.538975	0.597015	− 10.77
1	0.514231	0.571429	− 11.12
1.5	0.420055	0.470588	− 12.03
2	0.356572	0.400000	− 12.18
2.5	0.310525	0.347826	− 12.01
3	0.275438	0.307692	− 11.71
3.5	0.247735	0.275862	− 11.35
4	0.225263	0.250000	− 10.98
4.5	0.206641	0.228571	− 10.61
5	0.190941	0.210526	− 10.26
10	0.109304	0.117647	− 7.63
15	0.076912	0.081633	− 6.14
20	0.059422	0.062500	− 5.18
25	0.048448	0.050633	− 4.51
30	0.040913	0.042553	− 4.01
35	0.035415	0.036697	− 3.62
40	0.031225	0.032258	− 3.31
45	0.027925	0.028777	− 3.05
50	0.025258	0.025974	− 2.83
500	0.002646	0.002660	− 0.51

Defining

$$y = \frac{l}{2R} = \frac{l}{d}$$

we obtain

$$\alpha = 1 + y^2 - y\sqrt{y^2+1} - \frac{[(2-y^2)\sqrt{y^2+1} + y^3 - 2]^2}{4.5y\sqrt{y^2+1} - 4.5\ln[y + \sqrt{y^2+1}]}. \tag{2.24}$$

This gives results which agree with the Cole [11] data to within 0.13%.

Santeler [13] devised a simpler and more convenient formulation and calculates the transmission probability as

$$\alpha = \frac{1}{1 + \frac{3}{8}\frac{l_e}{R}}, \tag{2.25}$$

where l_e is an "equivalent length" and

$$\frac{l_e}{l} = 1 + \frac{1}{3 + \frac{3}{7}\frac{l}{R}}.$$

This gives transmission probabilities with a maximum error of less than 0.7% relative to the Cole data.

2.2.5. Duct of Uniform Rectangular Cross Section

The convention used to denote dimensions of rectangular (and elliptical) ducts is as follows: a and b are the cross-sectional dimensions, with $b \geq a$, and l is the length in the direction of gas flow. Thus the cross-sectional area is $A = ab$. To avoid any confusion, equations quoted from various authors have been recast to conform with this convention.

Long Ducts. The transmission probability, due to Smoluchowski [10], is

$$\alpha_r = \frac{a}{l}\left[\frac{\ln(\delta + \sqrt{1+\delta^2})}{\delta} + \ln\left(\frac{1+\sqrt{1+\delta^2}}{\delta}\right) + \frac{1 + \delta^3 - (1+\delta^2)^{3/2}}{3\delta^2}\right], \tag{2.26}$$

where $\delta = a/b$ and $l \gg b$.

A useful approximation is

$$\alpha_r = \frac{16}{3\pi^{3/2}}\frac{a}{l}\ln\left(4\frac{b}{a} + \frac{3}{4}\frac{a}{b}\right). \tag{2.27}$$

This is accurate to better than 1% for aspect ratios (b/a) up to almost 100. The error increases slowly with aspect ratio but is still only 1.9% and 2.4% for aspect ratios of 1000 and 10,000, respectively.

Short Ducts. There appear to be no general expressions which cover the whole range of lengths and aspect ratios. Data are available from the Monte Carlo calculations of Levenson et al. [14]. The results of Santeler and Boeckmann [15], which cover a greater range of lengths and aspect ratios, are shown in Table 2.6 (for short ducts the original data are listed to six significant figures). Also shown for comparison in Table 2.7 are some of the results of Cole [16], derived using a complementary variational method which gave upper and lower bounds for α; the values listed are the means and are accurate to 1.2% or better.

Table 2.6. Transmission Probabilities for Rectangular Ducts[a]

	b/a									
l/a	1	1.5	2	3	4	6	8	12	16	24
0.01	0.9902	0.9918	0.9926	0.9934	0.9938	0.9942	0.9944	0.9946	0.9947	0.9948
0.02	0.9807	0.9839	0.9854	0.9870	0.9878	0.9885	0.9889	0.9893	0.9895	0.9897
0.04	0.9626	0.9685	0.9715	0.9744	0.9759	0.9774	0.9782	0.9789	0.9793	0.9797
0.07	0.9370	0.9467	0.9515	0.9564	0.9589	0.9613	0.9625	0.9638	0.9635	0.9650
0.1	0.9131	0.9260	0.9326	0.9392	0.9425	0.9458	0.9475	0.9491	0.9500	0.9508
0.2	0.8428	0.8645	0.8757	0.8869	0.8926	0.8982	0.9011	0.9039	0.9053	0.9067
0.4	0.7334	0.7659	0.7829	0.8004	0.8093	0.8182	0.8227	0.8272	0.8295	0.8317
0.7	0.6178	0.6575	0.6793	0.7022	0.7140	0.7260	0.7321	0.7381	0.7411	0.7442
1	0.5363	0.5786	0.6026	0.6285	0.6421	0.6560	0.6631	0.6702	0.6737	0.6773
2	0.3780	0.4192	0.4444	0.4733	0.4893	0.5063	0.5150	0.5240	0.5285	0.5330
4	0.2424	0.2759	0.2977	0.3245	0.3404	0.3583	0.3679	0.3781	0.3833	0.3885
7	0.1596	0.1848	0.2020	0.2242	0.2380	0.2545	0.2639	0.2742	0.2796	0.2852
10	0.1195	0.1397	0.1537	0.1723	0.1843	0.1991	0.2078	0.2177	0.2230	0.2287
20	0.0655	0.0776	0.0864	0.0984	0.1066	0.1171	0.1238	0.1319	0.1366	0.1419
40	0.0346	0.041	0.0464	0.053	0.058	0.0652	0.0695	0.075	0.078	0.083
70	0.020	0.024	0.0275	0.032	0.035	0.039	0.042	0.046	0.048	0.052
100	0.014	0.017	0.019	0.023	0.025	0.028	0.030	0.033	0.035	0.038

[a] From Santeler and Boeckmann [15].

Table 2.7. Transmission Probabilities for Rectangular Ducts[a]

	b/a							
l/a	1	5	10	20	50	100	1000	10000
1	0.53619		0.66722			0.68266		
4	0.24233							
10	0.11930		0.21280			0.2372	0.2400	
20		0.11207						
40			0.07234					
80				0.04464				
100	0.01438		0.0320			0.0438	0.0464	0.0468
200					0.0224			
400						0.01295		

[a] From Cole [16].

For short slits (or large flat plates): Values of transmission probability were first calculated by Clausing [6]. Berman [12] devised equations for the direct calculation of transmission probability to a greater accuracy than the tabulated Clausing values. For $b \gg a$ and $b \gg l$ and putting $x = l/a$ we obtain

$$\alpha_r = 0.5[1 + \sqrt{1+x^2} - x] - \frac{1.5[x - \ln(x + \sqrt{1+x^2})]^2}{x^3 + 3x^2 + 4 - (x^2+4)\sqrt{1+x^2}}. \tag{2.28}$$

If $l \gg a$ (long, closely spaced slot, again with $b \gg a$, $b \gg l$), this equation simplifies to

$$\alpha_r \approx \frac{a}{l}\left[\ln\left(\frac{2l}{a}\right) - \frac{1}{2}\right]. \tag{2.29}$$

2.2.6. Uniform Elliptical Cross Section

a and b are the minor and major axes, $b \geq a$ $l \gg b$:

$$A = \frac{\pi}{4}ab.$$

Long Ducts. The expression derived from Eq. (2.18) by Steckelmacher [8] is

$$\alpha_e = \frac{8a}{3\pi l}K(k^2), \tag{2.30}$$

where

$$K(k^2) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}$$

is the complete elliptic integral, and

$$k^2 = 1 - \frac{a^2}{b^2}.$$

Steckelmacher [8] has shown that, for the same aspect ratio and cross-sectional area, the expressions for a rectangular duct (Eq. (2.26)] and an elliptical duct are in close agreement. The rectangular duct approximation was derived on this basis, so that a similar approximation is available for the elliptical duct:

$$\alpha_e = 0.856\frac{a}{l}\ln\left(4\frac{b}{a} + \frac{3}{4}\frac{a}{b}\right). \tag{2.31}$$

The constant is chosen to give the correct result for $b = a$ (circular) and minimises the errors for practical aspect ratios up to 10 (although the errors are less than 1% even for unrealistically large aspect ratios up to 1000).

Generally for ducts which have cross sections intermediate between rectangular and elliptical we have

$$\alpha_s \approx r_s \alpha_r, \tag{2.32}$$

where α_r is the rectangular duct transmission probability and

$$r_s = \sqrt{\frac{A_s}{a_s b_s}}. \tag{2.33}$$

Short Ducts. No data are available for short ducts of elliptical or similar cross section. However, it is expected that the similarity between long elliptical and rectangular ducts (of the same cross-sectional area and aspect ratio) will also apply to short ducts. It is suggested that approximate transmission probabilities can be found from

$$\alpha_s = r_s \left(\frac{1 + l/a}{r_s + l/a} \right) \alpha_r. \tag{2.34}$$

For very short ducts this reduces to $\alpha_s = \alpha_r$ since the transmission probability of an aperture is independent of shape.

2.2.7. Cylindrical Annulus (Flow Between Concentric Cylinders)

d_2 = outer diameter, d_1 = inner diameter,

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4} d_2^2 (1 - e^2), \qquad e = d_1/d_2.$$

Long Ducts

$$\alpha_a = \frac{d_2}{l} X(e),$$

$$X(e) = \frac{1}{1 - e^2}\left[\frac{4}{3} - e + e^3 - \frac{2}{3}(1 + e^2)\, E(e^2) + \frac{2}{3}(1 - e^2)\, K(e^2) \right], \tag{2.35}$$

where $K(e^2)$ and $E(e^2)$ are the complete elliptic integrals of the first and second kinds. $X(e)$ is listed for a range of values of e in Table 2.8.

Short Ducts. Table 2.9 presents the results of Berman [17], who calculated transmission probabilities over a large range using the variational method. Berman also obtained an empirical expression which is more convenient for computer calculation.

Table 2.8. The Function $X(e)$ for a Long Cylindrical Annulus [Eq. (2.31)]

$e = d_1/d_2$:	0	0.1	0.2	0.3	0.4	0.5
$X(e)$:	1.3333	1.231	1.1238	1.0116	0.8942	0.7711
$e = d_1/d_2$:	0.6	0.7	0.8	0.9	0.95	1.0
$X(e)$:	0.6416	0.5044	0.3576	0.1966	0.1071	0

Defining $x = l/(d_2 - d_1)$, we obtain

$$\alpha_a = \frac{1}{1 + x\left[1 - 2u \tan^{-1}\left(\frac{2x}{v}\right)\right]} \tag{2.36}$$

with

$$u = \frac{0.0741 - 0.014e - 0.037e^2}{1 - 0.918e + 0.05e^2},$$

$$v = \frac{5.825 - 2.86e - 1.45e^2}{1 + 0.56e - 1.28e^2}.$$

The expression is valid in the range $0 \leq x \leq 50$ and $0 \leq e \leq 0.9$.

2.2.8. Uniform Triangular Section (Equilateral)

a = length of sides,

$$A = \frac{\sqrt{3}}{4} a^2 = 0.433a^2.$$

Long Ducts

$$\alpha_t = \frac{(\sqrt{3} \ln 3)}{2} \frac{a}{l}. \tag{2.37}$$

Short Ducts. Approximate transmission probabilities can be obtained using the entrance correction principle [Eq. (2.20)].

2.2.9. Other Shapes

Transmission probabilities for a number of geometries are shown in graphical form in Figs. 2.2 [18], 2.3, and 2.4. It is worth noting that the transmission probability for an

Table 2.9. Transmission Probabilities ($\times 10^6$) for Cylindrical Annulus"

Table 2.9. Transmission Probabilities ($\times 10^4$) for Cylindrical Annulus[a]

$y = l/(R_2 - R_1)$	R_1/R_2									
	0.1	0.2	0.25	0.4	0.5	0.6	0.75	0.8	0.9	0.95
0.5	8017	8022		8030		8037		8043		8046
1.0	6737	6754		6783		6808		6829		6842
1.5	5842	5867		5915		5958		5997		6020
2.0	5175	5206		5266	5295	5324	5365	5378		5413
2.5	4655	4690		4758		4826		4894	4926	4940
3.0	4237	4274		4348		4423		4501		4558
3.5	3893	3931		4007		4087		4174		4241
4.0	3604	3642	3661	3720	3761	3804	3872	3896		3972
5.0	3123	3181		3260		3347		3448	3507	3538
6.0	2791	2828		2906	2948	2994	3071	3100		3201
7.0	2513	2548		2625		2712		2820		2929
8.0	2286	2321	2339	2395	2436	2481	2559	2589		2704
9.0	2099	2132		2204		2288		2496		2515
10.0	1914	1973		2042	2081	2124	2200	2230	2304	2352
12.0					1819		1933			
14.0					1617					
15.0	1414	1440		1499		1569		1666	1740	1792
16.0			1381		1456		1559			
18.0					1325					
20.0					1216		1310		1404	
25.0	921.7	941.1		984.5		1038		1116	1180	1230
30.0									1019	
35									897	
40									802	
50	496.0	507.6		533.9		567.4		618.2		700.4
100	258.9	265.4		280.1		299.2		328.9		380.5
200	132.7	136.1		144.0		154.3		170.6		200.1
500	53.97	55.4		58.69		63.04		70		82.99
1000	27.15	27.88		29.56		31.77		35.34		42.09
10^4	2.733	2.807		2.978		3.204		3.57		4.273
10^5	0.2735	0.2809		0.298		0.3207		0.3575		0.4282

[a] From Berman [17].

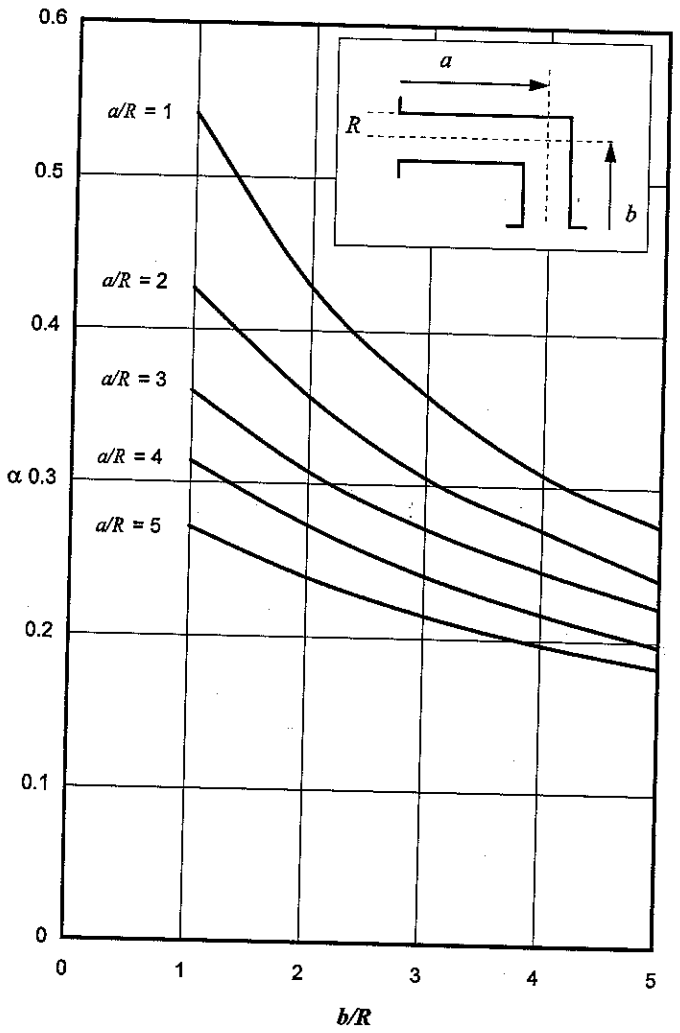

Fig. 2.2. Molecular transmission probabilities of an elbow, from the results of Davies [18].

elbow is almost the same as two short tubes (with length measured at the inside of the elbow) connected by a large volume. The effect of the elbow is to randomize, at least partly, the molecular distribution.

2.2.10. Combinations of Components

If two components, with transmission probabilities α_1 and α_2, are connected in series, then the usual method of determining the net transmission probability is

$$\frac{1}{\alpha_n} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}. \tag{2.38}$$

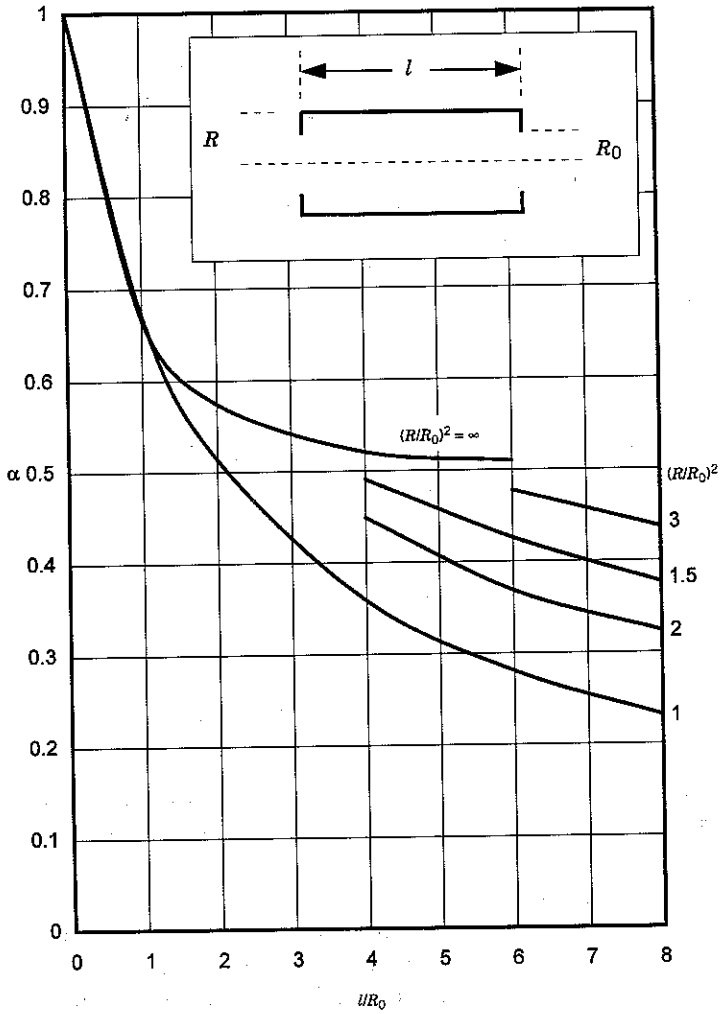

Fig. 2.3. Molecular transmission probabilities of a cylindrical tube with restricted openings, from the results of Davies [18].

Consider two identical, short tubes, of transmission probability α, connected via a large volume V as shown in Fig. 2.5*a*. The inlet of tube 2 and the outlet of tube 1 are also connected to large volumes and, for convenience, the downstream pressure is taken to be zero. It is supposed that there is no beaming between the tubes, and the effect of the large volume is to randomize the molecular distribution between the tubes.

The net flux of molecules through tube 2 is $(N_2 - N_1)\alpha$, and the net flux through tube 1 is $N_1\alpha$. Under steady conditions, these must be the same, so that $N_1 = N_2/2$ and the number of molecules transmitted is $N_2\alpha/2$. The overall transmission probability of the system is then $\alpha/2$. The conclusion is the same if the transmission probabilities are combined as in Eq. (2.38).

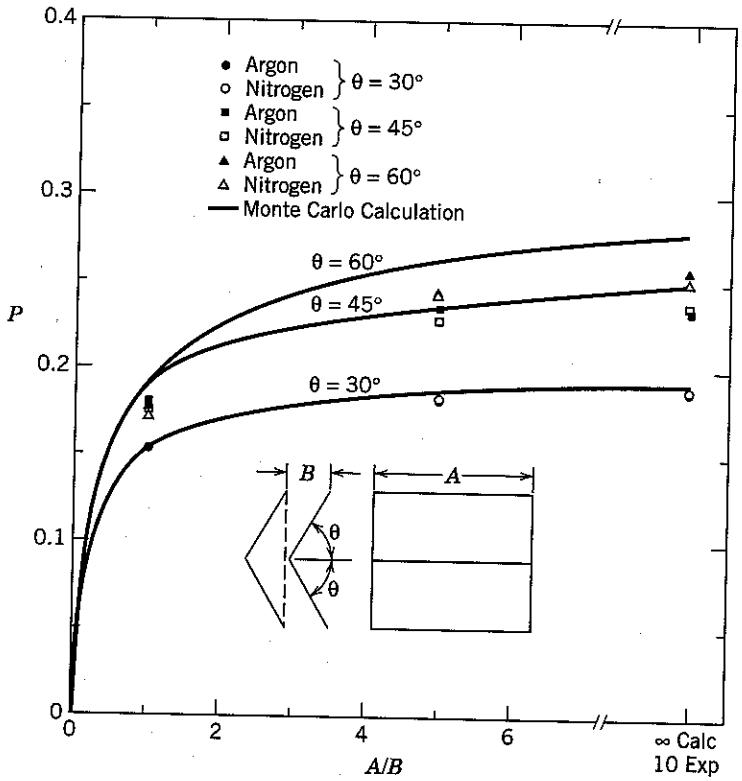

Fig. 2.4. Molecular transmission probabilities of a chevron baffle. Reproduced with permission from Levenson et al. [14] Copyright 1963, Société Française d'Engenieurs et Techniciens du Vide.

Now consider Fig. 2.5*b*, in which the two tubes (each with $l/d = 1$) have been brought together. The transmission probability of each tube separately is (from Table 2.5) 0.514. However, the transmission probability for the joined tubes ($l/d = 2$) is 0.357 and not $0.514/2 = 0.257$.

Clearly the method of combining transmission probabilities for the joined tubes is incorrect. In this case every molecule which crosses plane AA also crosses plane BB and vice versa, but this is not so when the tubes are separated by a large volume.

Oatley [19] discussed the correct method of combining transmission probabilities, α_1 and α_2, for two joined tubes of the same cross section and showed that the net transmission probability is given by

$$\frac{1}{\alpha_n} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} - 1. \tag{2.39}$$

In the example above, this gives the overall transmission probability as 0.346, which is much closer to the correct result. The Oatley method gives results with a maximum error of 5% or 6% for $l/d \sim 2$. It is, perhaps, surprising that the method gives such good results, since the derivation assumes random gas entry into the second tube and thus ignores the beaming effect. However, in short ducts the molecular distribution is

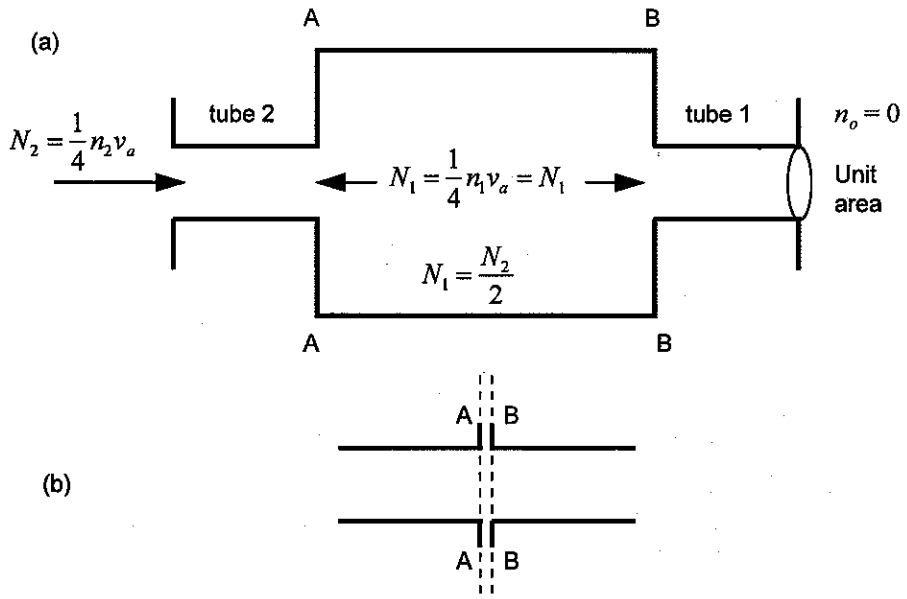

Fig. 2.5. Combination of two short tubes.

not too seriously perturbed from a chaotic distribution, and in long ducts the entrance effect is relatively small.

The transmission probabilities for each of the two tubes, in effect, includes an entrance correction. Thus, when Eq. (2.38) is applied, the overall transmission probability includes two entrance effects. This is correct when the two tubes are separated by a large volume but not when they are joined and the Oatley method is equivalent to removing one of the two corrections.

A typical case is illustrated in Fig. 2.6, in which a pump is connected to a chamber via a tube of the same size as the pump inlet. The pump speed (S) has been *measured*, so that any entrance effects are already accounted for (at least in principle). A pump can be regarded as a conductance with a transmission probability α_H equal to its Ho coefficient (the ratio of the pump speed to the conductance of the pump inlet aperture). If C_a is the pump and tube aperture conductance and α is the tube transmission probability, then $S = \alpha_H C_a$ and the tube conductance C equals αC_a. The net speed at the chamber is then $S_n = \alpha_n C_a$, where

$$\frac{1}{\alpha_n} = \frac{1}{\alpha_H} + \frac{1}{\alpha} - 1. \tag{2.40}$$

Alternatively,

$$\frac{1}{S_n} = \frac{1}{S} + \frac{1}{C_a}\left(\frac{1}{\alpha} - 1\right) = \frac{1}{S} + \frac{1}{C_m} - \frac{1}{C_a}. \tag{2.41}$$

The effect of this procedure is to remove an entrance correction.

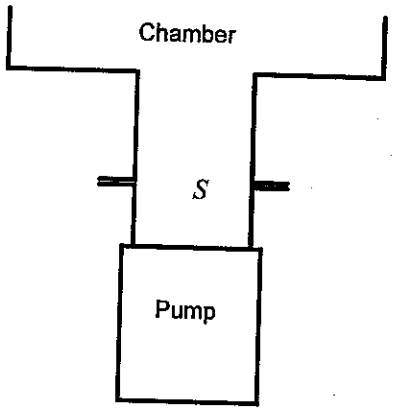

Fig. 2.6. Pump connected via a tube of the same diameter.

If the pump speed is 300 liter·s^{-1} and the connecting tube is 200 mm long and 100 mm in diameter, then $C_a = 908$ liter·s^{-1} (for air), α (for $l/d = 2$) $= 0.357$, and

$$\frac{1}{S_n} = \frac{1}{300} + \frac{1}{908}\left(\frac{1}{0.357} - 1\right),$$

which gives $S_n = 188$ liter·s^{-1} [instead of 155 liter·s^{-1} using Eq. (2.34)].

An addition theorem developed by Haefer [20] enables the calculation of multiple components of differing diameters. The overall transmission probability of n elements α_{1n} is related to the individual transmission probabilities α_i and inlet areas A_i by

$$\frac{1}{A_1}\left(\frac{1}{\alpha_{1n}} - 1\right) = \sum_1^n \frac{1}{A_i}\left(\frac{1}{\alpha_i} - 1\right) + \sum_1^{n-1}\left(\frac{1}{A_{i+1}} - \frac{1}{A_i}\right)\delta_{i,i+1} \tag{2.42}$$

with

$$\delta_{i,i+1} = 1 \quad \text{for } A_{i+1} < A_i \text{ (reducing cross-sectional area)}$$

$$\delta_{i,i+1} = 0 \quad \text{for } A_{i+1} \geq A_i \text{ (no reduction in cross-sectional area).}$$

The overall transmission probability α_{1n} is expressed in terms of the inlet tube aperture.

Some cases of the application of this theorem will be discussed with reference to Fig. 2.7.

***(a) Series Arrangement of Tubes of Different Diameters* (Fig. 2.7*a*).** The overall transmission probability is

$$\frac{1}{\alpha_{1n}} = \frac{1}{\alpha_1} + \frac{A_1}{A_2}\left(\frac{1}{\alpha_2} - 1\right) + \frac{A_1}{A_3}\left(\frac{1}{\alpha_3} - 1\right) + \left(\frac{A_1}{A_2} - 1\right)\delta_{2,1} + \left(\frac{A_1}{A_3} - \frac{A_1}{A_2}\right)\delta_{3,2}. \tag{2.43}$$

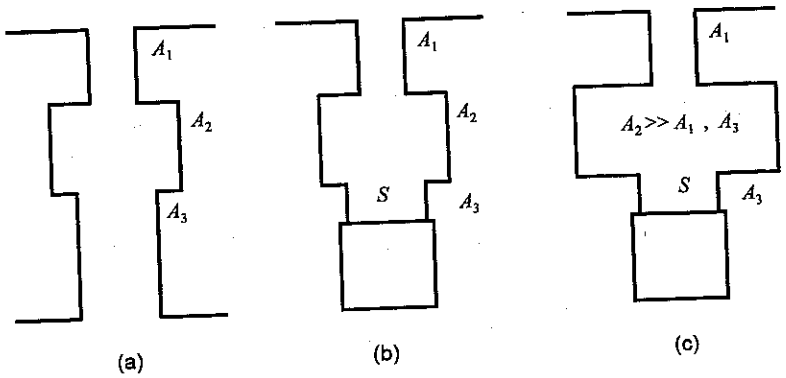

Fig. 2.7. Combinations of components.

The diameter increases from tube 1 to tube 2 so that $\delta_{2,1} = 0$, but $\delta_{3,2} = 1$ since the diameter decreases from tubes 2 to 3. If $l/d = 2$ for each tube and the diameters are 15 mm, 25 mm, and 20 mm for tubes 1, 2, and 3 respectively, then α_{1n} evaluates to 0.214. The aperture conductance of tube 1 (15-mm diameter) is 20.4 liter $\cdot$ s^{-1}, so the conductance of the arrangement is $0.214C_{a1} = 4.38$ liter $\cdot$ s^{-1}. If the order of the tubes is reversed (20 mm, 25 mm, 15 mm), α_{1n} now evaluates to 0.121. The inlet aperture conductance (for 20-mm diameter) is now 36.3 liter $\cdot$ s^{-1}, so the overall conductance is $0.121 \times 36.3 = 4.38$ liter $\cdot$ s^{-1} as expected. The conductance of an arrangement cannot be changed by reversing the order of components; but note that if the tubes were rearranged in ascending order of size, then $\alpha_{1n} = 0.224$. It is generally the case that the highest conductance for a series of components is achieved when they are physically arranged in order of size. In a calculation, the temptation to reorder the components for mathematical convenience should be resisted since this can lead to incorrect results.

Equation (2.42) relating transmission probabilities can be expressed in terms of conductances:

$$\frac{1}{C_{1n}} = \frac{1}{C_{a1}} + \sum_{1}^{n}\left(\frac{1}{C_{mi}} - \frac{1}{C_{ai}}\right) + \sum_{1}^{n-1}\left(\frac{1}{C_{ai+1}} - \frac{1}{C_{ai}}\right)\delta_{i,i+1}, \tag{2.44}$$

which is more convenient if conductance values are given for components. δ has the same meaning as in Eq. (2.42).

The first summation in Eq. (2.44) contains the "tube only" conductance C_{mt} discussed by Holland et al. [21]:

$$C_{mt} = \left(\frac{1}{C_m} - \frac{1}{C_a}\right); \tag{2.45}$$

that is, a tube with its entrance correction subtracted. If the conductance quoted for a component is the "tube only" value then the entrance correction should not be subtracted. A typical example would be a quarter swing valve which is not designed for direct connection to a large chamber but is normally connected via a manifold. Similar remarks apply to Eq. (2.42) if the tube only transmission probability is given.

Either of Eqs. (2.42) or (2.44) can be used to find the net speed of a pump in series with a set of components.

***(b) Pump Connected to a Chamber Via Two Tubes or Components* (Fig. 2.7*b*)**

$$\frac{1}{S_n} = \frac{1}{C_{a1}} + \left(\frac{1}{C_{m1}} - \frac{1}{C_{a1}}\right) + \left(\frac{1}{C_{m2}} - \frac{1}{C_{a2}}\right) + \left(\frac{1}{C_{m3}} - \frac{1}{C_{a3}}\right) + \left(\frac{1}{C_{a3}} - \frac{1}{C_{a2}}\right). \tag{2.46}$$

Since $C_{m3} = \alpha_H C_{a3} = S$, this reduces to

$$\frac{1}{S_n} = \frac{1}{C_{m1}} + \frac{1}{C_{m2}} + \frac{1}{S} - \frac{2}{C_{a2}}, \tag{2.47}$$

which can be written

$$\frac{1}{S_n} = \left(\frac{1}{C_{m1}} - \frac{1}{C_{a1}}\right) + \left(\frac{1}{C_{a1}} - \frac{1}{C_{a2}}\right) + \left(\frac{1}{C_{m2}} - \frac{1}{C_{a2}}\right) + \frac{1}{S}. \tag{2.48}$$

Written in terms of "tube only" conductances, this becomes

$$\frac{1}{S_n} = \frac{1}{C_{mt1}} + \left(\frac{1}{C_{a1}} - \frac{1}{C_{a2}}\right) + \frac{1}{C_{t2}} + \frac{1}{S}. \tag{2.49}$$

The second term on the right-hand side of this equation can be seen as the total correction required to account for the differing sections (remembering that a correction for A_3 is inherent in the pump speed).

If, in Fig. 2.7*b*, tubes 1 and 2 and the pump all have the *same* aperture size of conductance C_a, then Eq. (2.48) becomes

$$\frac{1}{S_n} = \left(\frac{1}{C_{m1}} - \frac{1}{C_a}\right) + \left(\frac{1}{C_{m2}} - \frac{1}{C_a}\right) + \frac{1}{S}. \tag{2.50}$$

Each tube has its entrance correction subtracted and only one correction, inherent in the pump speed, is applied.

***(c) Pump Connected to a Chamber Via a Second Chamber* (Fig. 2.7*c*).** This case is equivalent to making C_{a2} and C_{m2} very large in Eq. (2.47), giving

$$\frac{1}{S_n} = \frac{1}{C_{m1}} + \frac{1}{S}. \tag{2.51}$$

In this case the entrance correction for tube 1 is retained.

2.2.11. *Cases of Unsteady Flow.* Consider a system in which a pump is connected to a vessel via a valve and a pipe of some significant length. It is clear that, at the instant that the valve is opened, the throughputs at the pump and vessel must be

different. Some time is required for continuity to be established. Usually this is very short compared with the timescale for exhaust of the vessel, especially at continuum flow pressures where the pipe conductance is relatively high. However, if the "vessel" is a long pipe, then steady conditions are never achieved.

Unsteady flow is of most interest under molecular flow conditions and is particularly relevant to filling, exhaust, or leak testing of long pipelines. Mathematically, unsteady molecular flow in a constant section duct is analogous to heat conduction in an infinite slab. Both can be treated as one-dimensional since there is no transverse flow. The cases described here are intended only as a small sample to illustrate the kind of unsteady flow problems that can be solved. Solutions for many heat conduction cases, which have practical parallels in molecular flow, are available in the literature; Carslaw and Jaeger [22] is a good source of reference.

In the equations which follow, V is the volume and C the conductance of the pipe, assumed to be long. Given a relation for pressure distribution, the throughput (from or into the pipe) can be obtained from the pressure gradient at the end of a pipe:

$$\dot{Q} = Cl\left(\frac{dP}{dx}\right)_{\text{end}}. \tag{2.52}$$

Case 1. Pipe with uniform initial pressure P_0, closed at $x = 0$. $x = l$ opened to an environment maintained at constant pressure P_e at $t = 0$.

$$P - P_e = \frac{4}{\pi}(P_0 - P_e)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\exp\left\{-(2n+1)^2\frac{\pi^2}{4}\frac{C}{V}t\right\}\cos\left[(2n+1)\frac{\pi}{2}\frac{x}{l}\right]. \tag{2.53}$$

This covers both exhaust and filling of a pipe. Lawson [23] discussed the application of this and similar cases to pumping of trapped volumes and leakage.

The time constant for the first (slowest) term is

$$\tau = \tau_0 = \frac{4}{\pi^2}\frac{V}{C}. \tag{2.54}$$

Contrast this with the time constant of a chamber of volume V pumped at speed S or through a restrictive conductance C (covered in Chapter 9).

For air at 20°C (l and d in meters) we have

$$\tau_0 = 0.00263\frac{l^2}{d}\text{ sec}, \tag{2.55}$$

and for helium we have

$$\tau_0 \approx \frac{l^2}{d}\text{ msec}. \tag{2.56}$$

A 1-meter-long pipe, 10 mm in diameter would have a time constant for helium of ~100 msec. For a pipe length of 10 meters, this increases to ~10 sec. Leak testing can be difficult with a detector at the downstream end of such long lines since any leak will be located at the position of the helium probe several seconds prior to the indication on the leak detector gauge.

The series converges quite rapidly and taking only the first term is a good approximation for $t > 0.3\tau_0$. Thus, taking only the first term we obtain, at $x = 0$

$$P - P_e = \frac{4}{\pi}(P_0 - P_e)\exp\left(-\frac{\pi^2}{4}\frac{C}{V}t\right) \tag{2.57}$$

and the throughput

$$\dot{Q} = 2C(P_0 - P_e)\exp\left(-\frac{\pi^2}{4}\frac{C}{V}t\right). \tag{2.58}$$

Case 2. Pipe, with constant initial pressure P_0, closed at $x = 0$ and pumped at the other end by a pump of constant speed S, for $t > 0$.

$$P = P_0 2r \sum_{n=1}^{\infty} \frac{\cos\left(\phi_n \frac{x}{l}\right)\sec\phi_n}{r(r+1)+\phi_n^2}\exp\left(-\phi_n^2\frac{C}{V}t\right), \tag{2.59}$$

where $r = S/C$ and ϕ_n are the roots of

$$\phi\tan\phi = \frac{S}{C}. \tag{2.60}$$

A good approximation for the first root is

$$\phi_1^2 = \frac{S}{C + \frac{4}{\pi^2}S}, \tag{2.61}$$

accurate to within 3% in the worst case.

Taking only the first term of the series for throughput, we obtain

$$\dot{Q} = 2P_0 S\frac{\phi_1\tan\phi_1}{r(r+1)+\phi_1^2}\exp\left(-\phi_1^2\frac{C}{V}t\right). \tag{2.62}$$

For $S \ll C$ the time constant becomes V/S and the case reduces to the simple pumpout of a large chamber.

Case 3. Pipe with zero initial pressure. Constant leak $\dot{Q}_L$ into the pipe at $x = 0$ and constant pumping speed S at $x = l$ for $t > 0$.

This is again relevant to leak test of a long pipeline since the initial partial pressure of He will be zero.

$$P = \frac{\dot{Q}_L}{S}\left\{1 + r\left(1 - \frac{x}{l}\right) - 2r\sum_{n=1}^{\infty}\frac{(\phi_n^2 + r^2)\cos\left(\phi_n\frac{x}{l}\right)}{\phi_n^2[r(r+1)+\phi_n^2]}\exp\left(-\phi_n^2\frac{C}{V}t\right)\right\}, \tag{2.63}$$

where $r = S/C$ and ϕ_n has the same meaning as in Case 2.

Taking only the first term, the throughput from the pipe is

$$\dot{Q} = \dot{Q}_L\left\{1 - 2\left[\frac{(\phi_1^2 + r^2)\sin\phi_1}{\phi_1[r(r+1)+\phi_1^2]}\exp\left(-\phi_1^2\frac{C}{V}t\right)\right]\right\}. \tag{2.64}$$

2.3. CONTINUUM FLOW

In the continuum regime, calculation of gas flow through ducts is complicated by the different types of flow which can occur. Flow may be broadly distinguished into two major types, referred to as *viscous laminar flow* and *turbulent flow*. Since flow through a duct is driven by a pressure difference, all gas flow is compressible. There are circumstances in which gas can be treated as incompressible, and this leads to considerable simplification of the equations describing flow. However, there are also many circumstances in which compressibility cannot be ignored, so *compressible flow* will also be discussed.

Continuum (or viscous) flow is often thought of as occurring at relatively high pressures. But consider air flowing through a 100 mm (4 inch) diameter pipe. From Eq. (2.2) the Knudsen number Kn is < 0.01 (and hence the flow continuum) down to pressures of about 0.1 mbar. Thus many vacuum processes will operate at pressures where continuum flow conditions prevail.

At relatively low velocities, gas flows smoothly in stream lines, generally parallel to the duct walls, and the flow is said to be laminar. In long ducts, viscosity of the gas is a controlling factor in the flow rate; this is not the case in short ducts, although the flow may still be laminar. As the flow velocity is increased, there comes a critical point at which the flow breaks up into turbulent eddies. These two types of flow, viscous laminar and turbulent, are described by different equations. It is important to distinguish these flow types in calculation of flow rates; failure to do so can lead to wildly inaccurate results. In short ducts, or in longer ducts at high flow velocities, compressibility becomes important and use of incompressible flow formulae can also lead to serious errors.

The primary controlling parameter in the viscous behavior of Newtonian* fluids is the dimensionless Reynolds number

$$\mathrm{Re} = \frac{\rho u D_h}{\eta}, \tag{2.65}$$

*Newtonian fluids are those in which the shear stress is proportional to the transverse velocity gradient. Most common fluids (water, oils, and gases) are newtonian.