

PHYS 610: Recent Developments in Quantum Mechanics and Quantum Information
(Spring 2009)
Course Notes: The Density Operator and Entanglement

1 Multiple Degrees of Freedom

1.1 Merging Hilbert Spaces

Suppose two degrees of freedom are prepared in two quantum states completely independently of each other. This could happen, say, for two particles prepared in separate, distant galaxies. We will refer to the two degrees of freedom as “particles,” even though they could correspond to different degrees of freedom of the *same* system, such as the spin and center-of-mass position of an atom, or the spin and spatial profile of a photon.

Labeling the two particles as A and B , if the individual states of the particles are $|\psi\rangle_A$ and $|\psi\rangle_B$, then we can write the composite state as

$$|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B, \quad (1)$$

where \otimes denotes the **tensor product** (or **direct product**). Often, this product is written without an explicit tensor-product symbol:

$$|\psi\rangle_A \otimes |\psi\rangle_B \equiv |\psi\rangle_A |\psi\rangle_B \equiv |\psi_A \psi_B\rangle. \quad (2)$$

The particle labels can even be dropped, since the ordering determines which state applies to which particle.

We can also see the meaning of the tensor product in component form. Let each separate state be expressed in an orthonormal basis as

$$|\psi\rangle_A = \sum_{\alpha} c_{\alpha}^{(A)} |\alpha\rangle_A, \quad |\psi\rangle_B = \sum_{\beta} c_{\beta}^{(B)} |\beta\rangle_B. \quad (3)$$

Then we can express the composite state as

$$|\psi\rangle = \sum_{\alpha\beta} c_{\alpha\beta} |\alpha_A \beta_B\rangle, \quad (4)$$

where

$$c_{\alpha\beta} = c_{\alpha}^{(A)} c_{\beta}^{(B)}. \quad (5)$$

Note that $c_{\alpha\beta}$ is still understood to be a *vector*-like object, with a single index. Thus, there is an implicit (bijective) mapping of the ordered index pair (α, β) to a single index, which we simply denote as $\alpha\beta$.

Similarly, we can write a density operator for two independent particles by the same tensor product:

$$\rho = \rho^{(A)} \otimes \rho^{(B)}. \quad (6)$$

We can also write this in component form for the density matrices as

$$\rho_{\alpha\mu\beta\nu} = \rho_{\alpha\beta}^{(A)} \rho_{\mu\nu}^{(B)}, \quad (7)$$

where again $\alpha\mu$ and $\beta\nu$ are to be taken as composite indices.

The same tensor-product notation applies to Hilbert spaces. That is, we can write

$$|\psi_A \psi_B\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \quad (8)$$

if $|\psi\rangle_A \in \mathcal{H}_A$ and $|\psi\rangle_B \in \mathcal{H}_B$.

1.2 Entanglement

The above composite states, described by tensor products of separated states, are called **separable states**. However, not all states are separable, and those that are not separable are called **entangled**. In some sense, entanglement is the “most quantum” of all quantum effects.

Thus, we can see that a composite state $|\psi\rangle$ is entangled if and only if it *cannot* be written in the separable form

$$|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B. \quad (9)$$

The definition for density operators is somewhat more general: a density operator for a composite system is separable if and only if it can be written in the form

$$\rho = \sum_{\alpha} P_{\alpha} \rho_{\alpha}^{(A)} \otimes \rho_{\alpha}^{(B)}. \quad (10)$$

Unfortunately, given an arbitrary mixed density operator, it is difficult to tell if it corresponds to an entangled state (in fact, this turns out to be an NP-hard problem).

The point is that two entangled systems do not have local states that can be treated independently. This is in conflict with the apparently reasonable assumption of **local realism**, which states that distant systems should have independent, observer-independent realities (in particular, they should not directly influence each other). Herein lies the importance of the famous Bell inequalities and their experimental verification: local realism contradicts quantum mechanics, and so we must either give up locality or realism. Most modern practitioners of quantum mechanics choose to give up realism, which says that systems have observer-independent realities, in favor of locality. The Bohm formulation of quantum mechanics is a well-known realistic (but nonlocal) theory.

1.2.1 Cloning

With the language of entanglement, it is relatively simple to demonstrate the **no-cloning theorem**,¹ which says that the state of a *single* quantum system cannot be copied to another particle. This turns out to be a simple consequence of unitary evolution.

Let’s examine just a simple case. Suppose that cloning is possible on a two-state system from particle A to particle B . Particle B must be in a particular state to begin with, and without loss of generality we may take this to be the “0” state. Then to copy the eigenstates of A , we see that there must be a unitary transformation U such that

$$U|0\rangle_A|0\rangle_B = |0\rangle_A|0\rangle_B, \quad U|1\rangle_A|0\rangle_B = |1\rangle_A|1\rangle_B. \quad (11)$$

However, if particle A is in the superposition state

$$|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A), \quad (12)$$

then we see that the cloning operator gives

$$U|\psi\rangle_A|0\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B), \quad (13)$$

¹W. K. Wootters and W. H. Zurek, “A single quantum cannot be cloned,” *Nature* **299**, 802 (1982); D. Dieks, “Communication by EPR devices,” *Physics Letters A* **92**, 271 (1982).

which is the entangled **Schrödinger-cat state**. However, what we wanted for cloning to work properly is the *separable* state

$$U|\psi\rangle_A|0\rangle_B = \frac{1}{2}\left(|0\rangle_A + |1\rangle_A\right)\left(|0\rangle_B + |1\rangle_B\right). \quad (14)$$

We can see that the problem in this particular example is that U acts nonlocally, and thus induces entanglement between the two particles. In fact, the **controlled-NOT** (CNOT) gate is a quantum operation that effects the transformations in Eqs. (11) (if A and B are in eigenstates, the CNOT flips the state of system B if and only if system A is in state 1).

Of course, it *is* possible to clone a state if you already know everything about it (i.e., you have classical knowledge of the state), or if you have an infinite ensemble of copies. (Copying a state is possible to within some fidelity tolerance for a finite ensemble of copies.) In this case, enough measurements may be made to reconstruct the state of the system arbitrarily well, and of course this procedure does not correspond to a unitary transformation. The problem with the *single* system is that in general, a measurement of the system destroys its state, and a single measurement is not enough to determine the state of the system. Of course, there is no problem with the cloning of the *basis* states, as in Eqs. (11); the problem is in cloning general states that are not orthogonal to the basis states. In particular this means that with a bit of *extra* information beyond what is contained in the quantum state (e.g., the state of particle A is either $|0\rangle_A$ or $|1\rangle_A$, but not any coherent superposition of the two), cloning may in fact be possible.

1.3 Open Systems: Church of the Larger Hilbert Space

One important function use of the density operator is in describing *open* quantum systems—systems interacting with auxiliary systems (*environments* or *reservoirs*) that we don't have access to. We will treat open quantum systems in great detail, but for now let's examine a simple model for why the density operator is useful.

Consider the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle\right) \quad (15)$$

between particles (qubits) A and B . Suppose that we have access to particle A , but particle B is locked up in a box, so that we don't know anything about it. The density operator for the composite system is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\left(|0_A\rangle|0_B\rangle\langle 0_A|\langle 0_B| + |1_A\rangle|1_B\rangle\langle 1_A|\langle 1_B| + |1_A\rangle|1_B\rangle\langle 0_A|\langle 0_B| + |0_A\rangle|0_B\rangle\langle 1_A|\langle 1_B|\right) \quad (16)$$

We can define the **reduced density operator** that describes *only* particle A by performing a partial trace over the state of particle B :

$$\rho_A = \text{Tr}_B[\rho] := \sum_{\alpha_B} \langle\alpha_B|\rho|\alpha_B\rangle = \frac{1}{2}\left(|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|\right). \quad (\text{reduced density operator}) \quad (17)$$

Thus, we can see that the reduced state of particle A corresponds to a *completely incoherent* superposition of the two states, even though the composite system carried a completely coherent superposition.

This is a simple model for the process of **decoherence**. A quantum system can start in a local state of coherent superposition. But if it interacts with the environment, the coupling causes entanglement between the system and environment, since the interaction is nonlocal. Because we don't have access to the state of the environment, we must trace over it, which reduces the purity of the reduced density operator. Note that we can't keep track of the environment *even in principle*, since it generally has many degrees of freedom. As the interaction continues, the entanglement progresses, driving the reduced density operator towards a

completely incoherent superposition. This is, at a simple level, why classical (macroscopic) things behave classically: coupling to the environment destroys quantum coherence.

Conversely, whenever we have a system described by a mixed density operator,

$$\rho = \sum_{\alpha} P_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}|, \quad (18)$$

we can always think of it as part of a larger system. We can see this as follows. We will introduce a fictitious environment with orthonormal basis states $|\alpha_E\rangle$. Then we can write the state vector for the composite system as

$$|\psi_{\text{total}}\rangle = \sum_{\alpha} \sqrt{P_{\alpha}} |\psi_{\alpha}\rangle |\alpha_E\rangle. \quad (19)$$

(purification of mixed state)

When we compute the total density operator for the composite pure state and trace over the environment, we recover the original density operator (18) as the reduced density operator of the larger state. This procedure of switching to a larger pure state is referred to as **purification** or “the doctrine of the Church of the larger Hilbert space.”² The extra environment degree of freedom is often referred to as the *ancilla*. Often, this is a useful picture for thinking about mixed quantum states, especially in quantum-information problems.

²Terminology introduced by John Smolin; see Daniel Gottesman and Hoi-Kwong Lo, “From Quantum Cheating to Quantum Security,” *Physics Today* **53**, no. 11, 22 (2000). (Online link: <http://www.aip.org/pt/vol-53/iss-11/p22.html>.)